

# A Worst-Case Approximate Analysis of Peak Age-of-Information Via Robust Queueing Approach

**Zhongdong Liu**

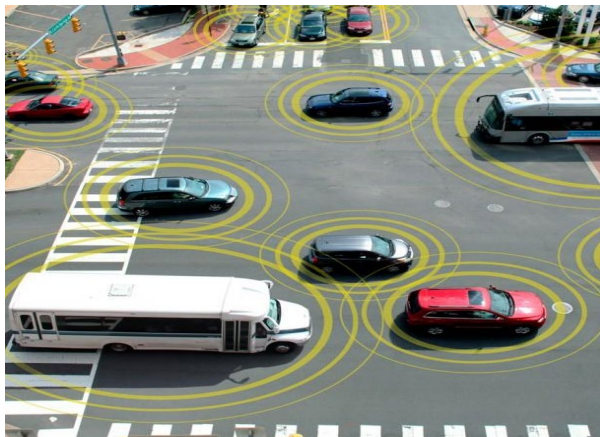
Department of Computer Science  
Virginia Tech

Joint work with

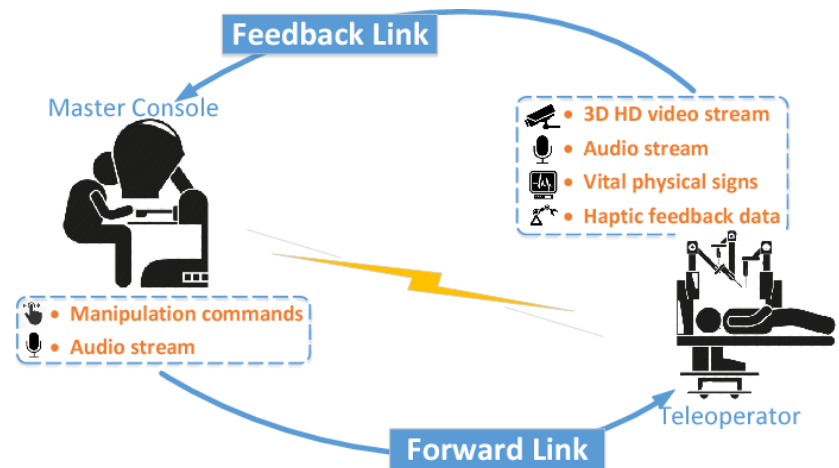
**Yu Sang** (Temple University), **Bin Li** (University of Rhode Island) and **Bo Ji** (Virginia Tech)

# Freshness Matters

- Real-time services are ubiquitous
  - Intelligent transportation systems & vehicular networks
  - Sensor networks (for environmental/health monitoring)
  - Wireless channel feedback, news feeds, weather updates, etc.



Intelligent vehicular networks



Sensor networks

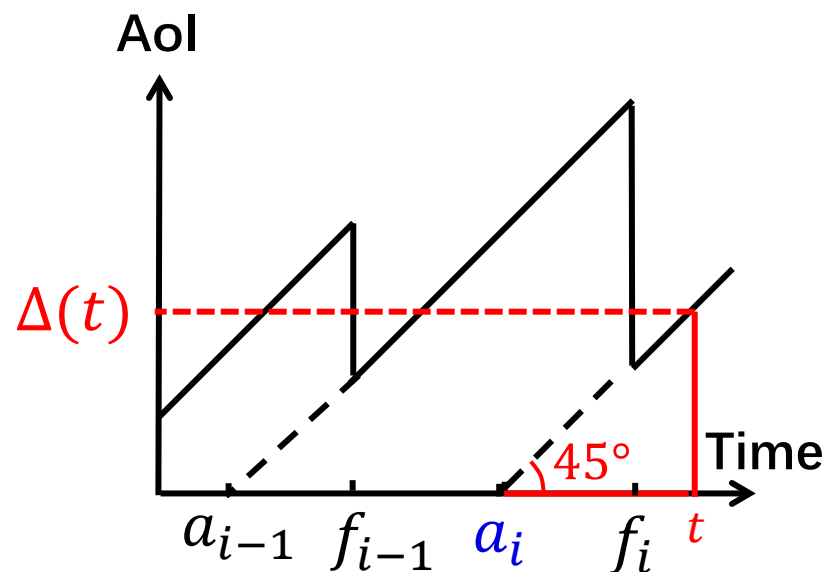
<https://www.networkworld.com/article/2881654/security0/wireless-cyber-security-in-your-car-stinks.html>

# Metric of Freshness

- **Age-of-information (Aol):** The time difference between current time and the generation time of the latest received update

If update  $i$  is generated at  $a_i$  and delivered at  $f_i$ , then Aol at time  $t$  is

$$\Delta(t) = t - \max\{a_i : f_i \leq t\}$$



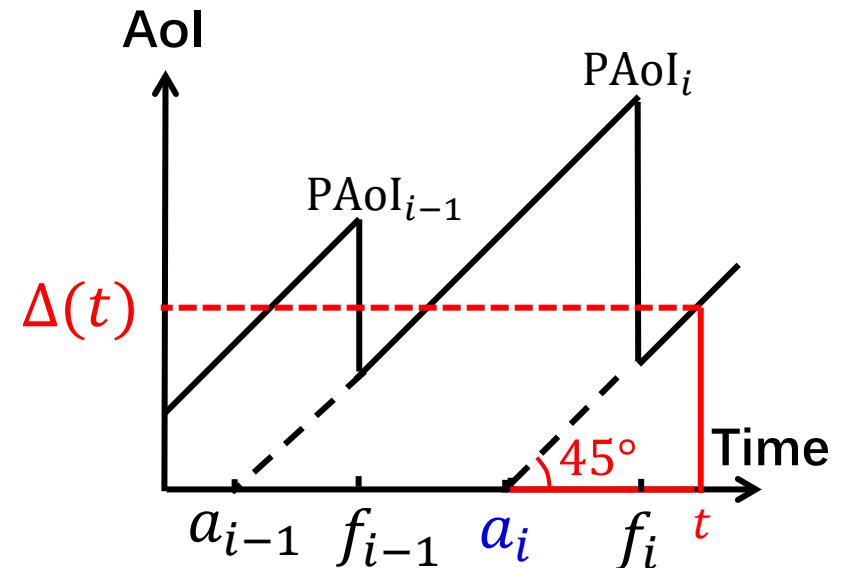
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- **Peak Age-of-Information (PAol):** The maximum value of the Aol before it drops



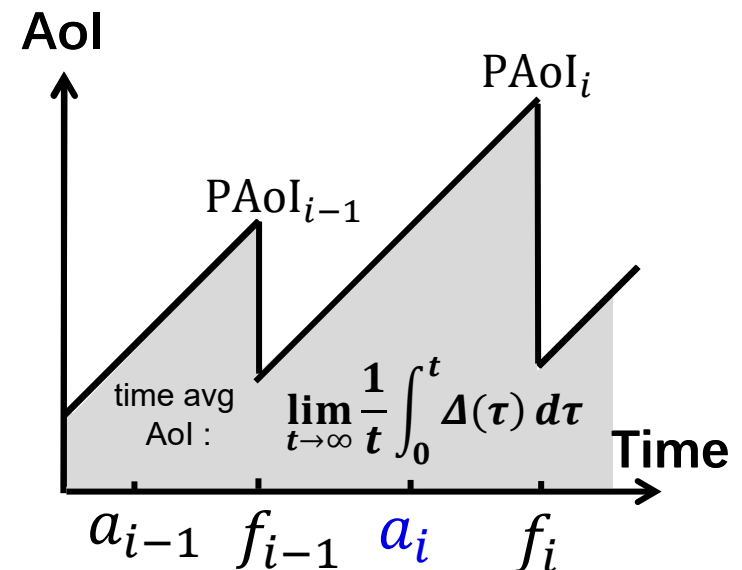
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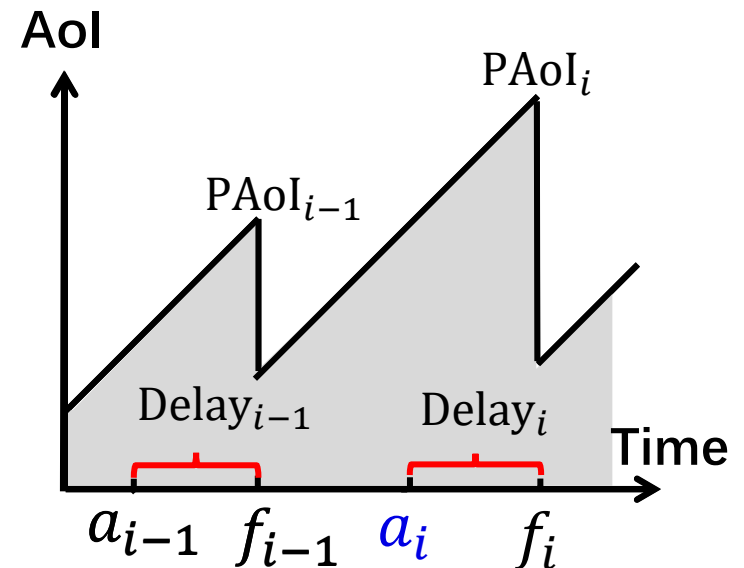
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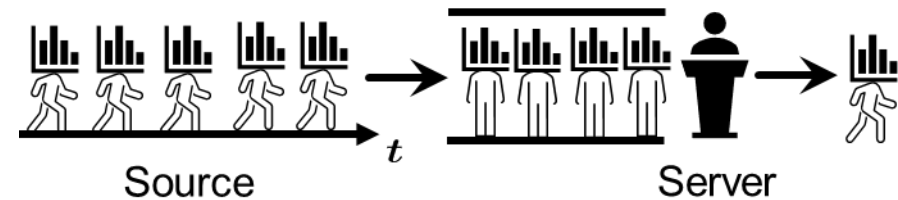
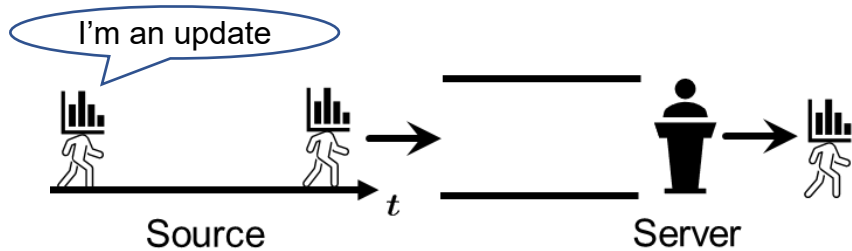
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- **Peak Age-of-Information (PAol):** The maximum value of the Aol before it drops
- **Delay:** The time difference between the generation time and the delivery time



# AoI vs. Delay

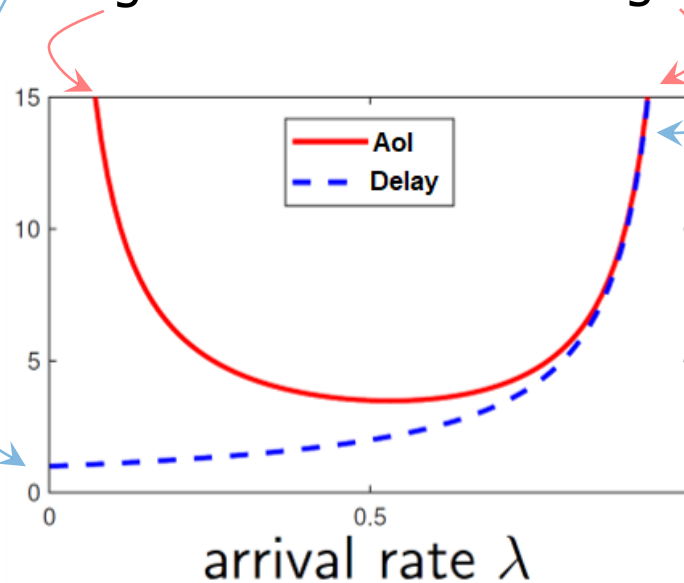


## ● Low arrival rate

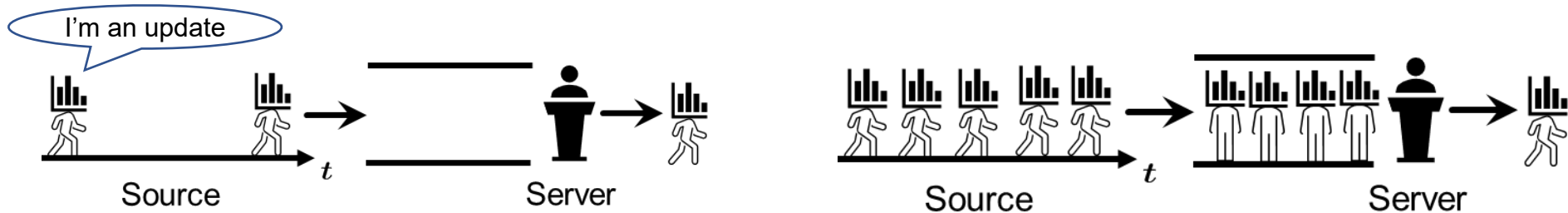
- Empty buffer → low delay
- Infrequent updates → long interarrival time & high AoI

## ● Large arrival rate

- Full buffer → high delay
- Become stale while waiting → high AoI

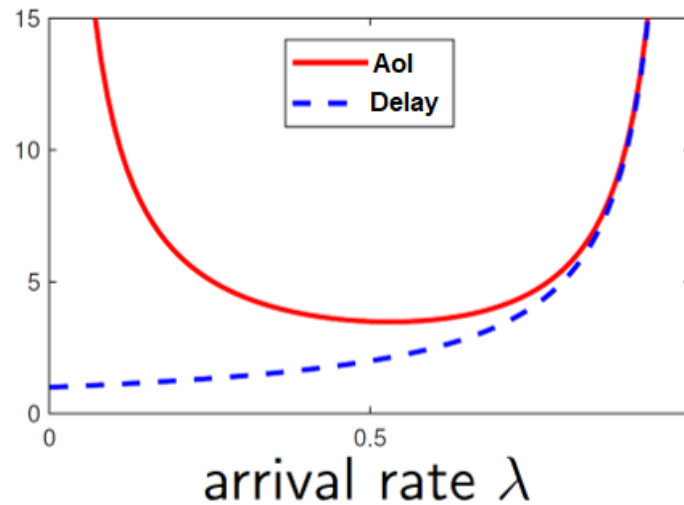


# Aol vs. Delay



Aol/PAol are large under both

- Low-load regime
- High-load regime





# Related Work

- Aol studies:
  - Assume **certain distributions** for arrival and service processes:
    - Aol in M/M/1, M/D/1, D/M/1 queues under the FCFS policy [Kaul et al. '12]
    - Aol in M/Gamma/1 queue under the PLCFS policy [Najm et al. '16]
  - Assume **i.i.d. arrival and service process**:
    - Aol in GI/GI/1, M/GI/1, and GI/M/1 queues [Inoue et al. '19]
- Delay studies:
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  - Only focus on **high-load regime**:
    - Robust queueing approach [Bandi et al. '15]

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practical?

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## Our contributions:

- Applying the **robust queueing theory** to analyzing **PAol** performance

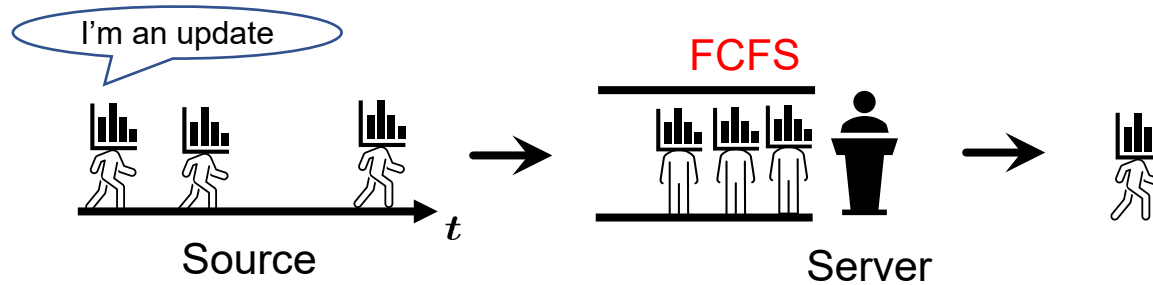
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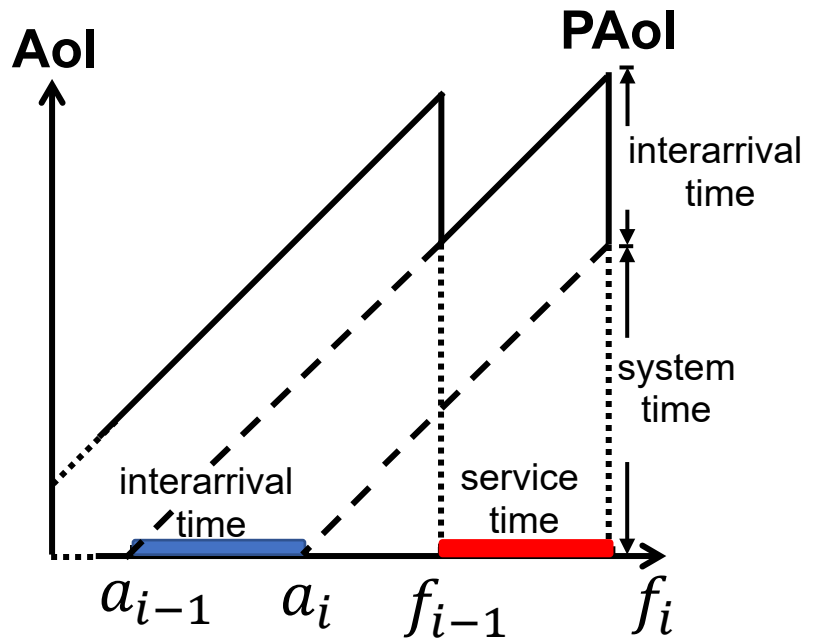
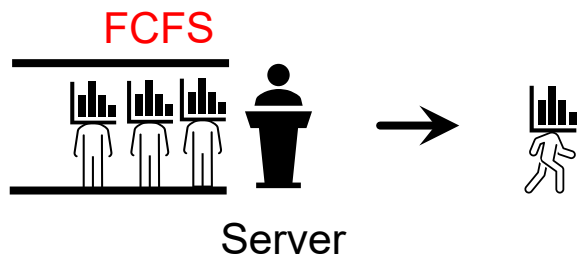
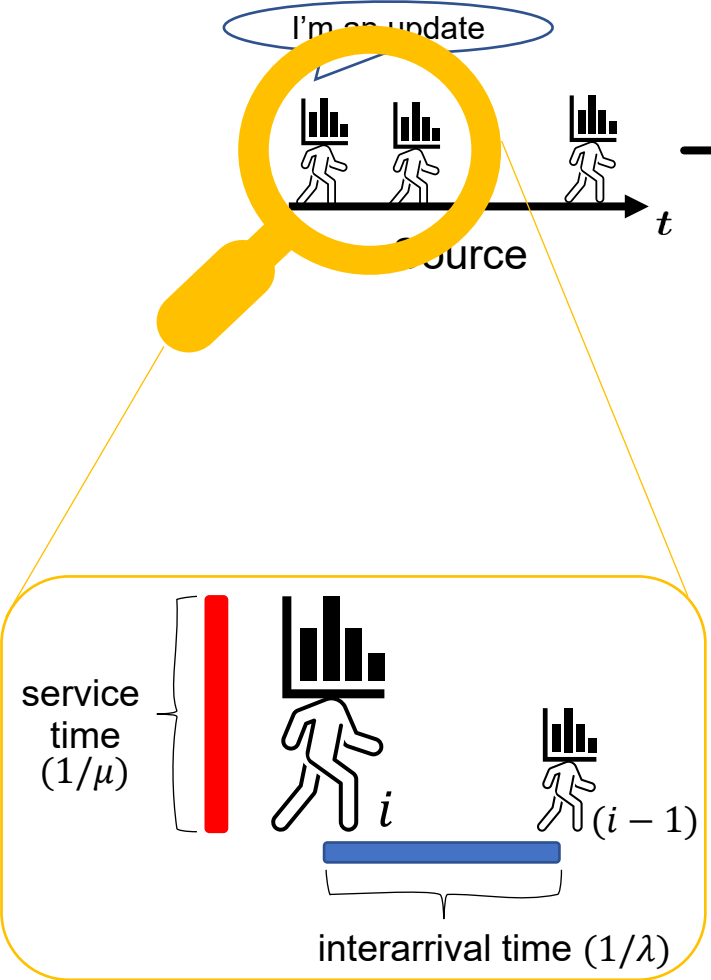
## Our contributions:

- Applying the **robust queueing theory** to analyzing **PAol** performance
- Approximating expected PAol well under **both high and low load regimes**

# Single-Source System (G/G/1)

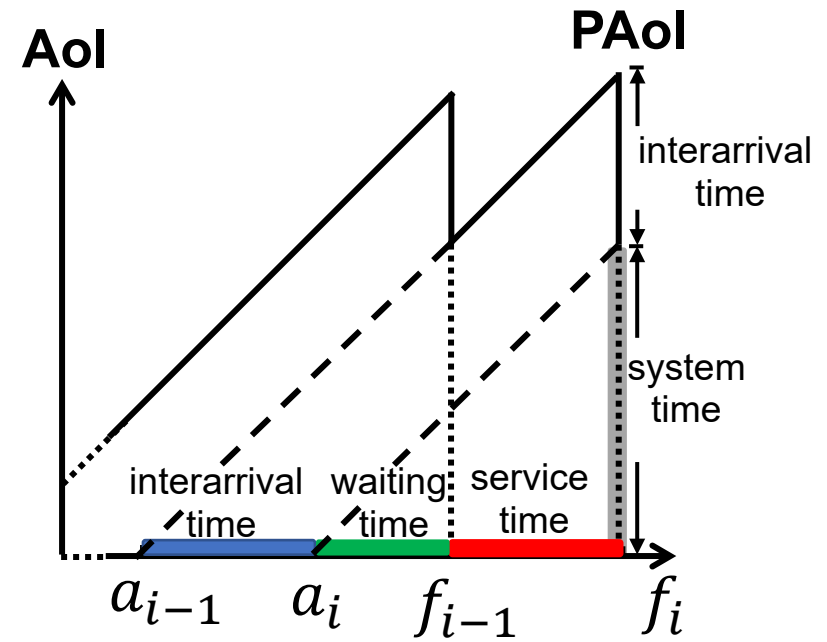
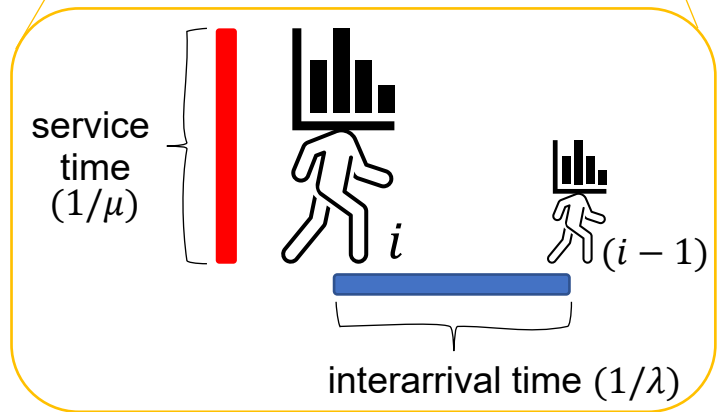


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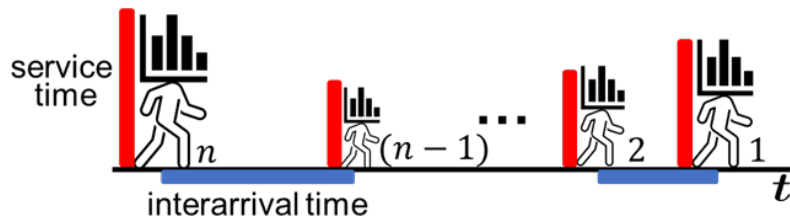


# Single-Source System (G/G/1)



# The Robust-Queueing Approach

## Uncertainty sets of arrival/service process

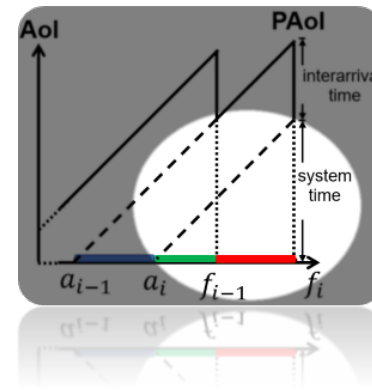


- Modeling their stochastic properties



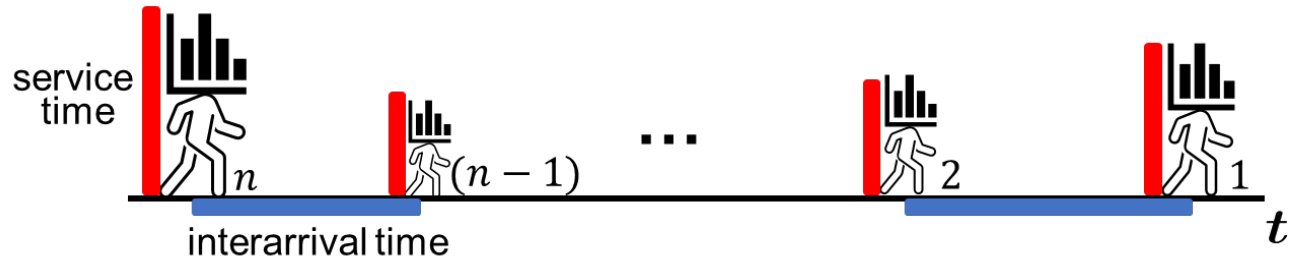
- Getting rid of **unrealistic assumptions** (e.g., memoryless properties or *i.i.d.*)

## Worst-case analysis of system time

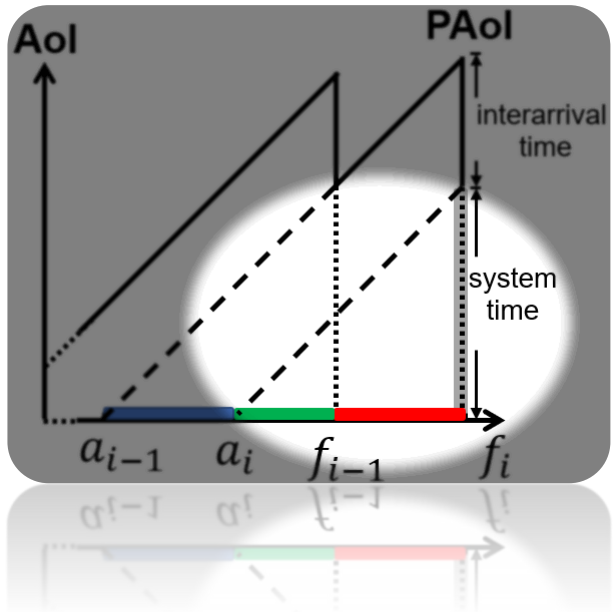
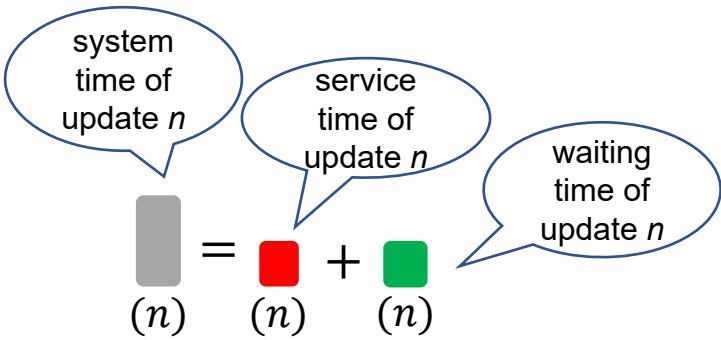
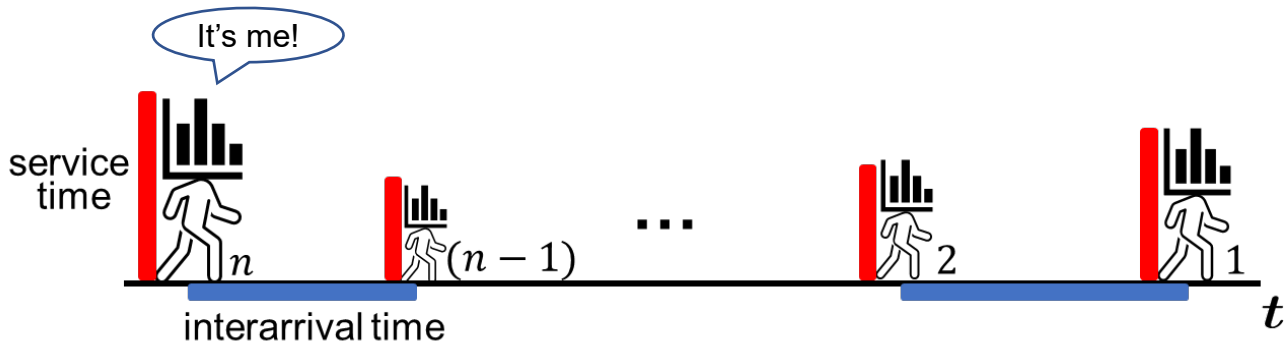


- Used to approximate the expected PAol
- Work well under **both low-load and high-load regimes**

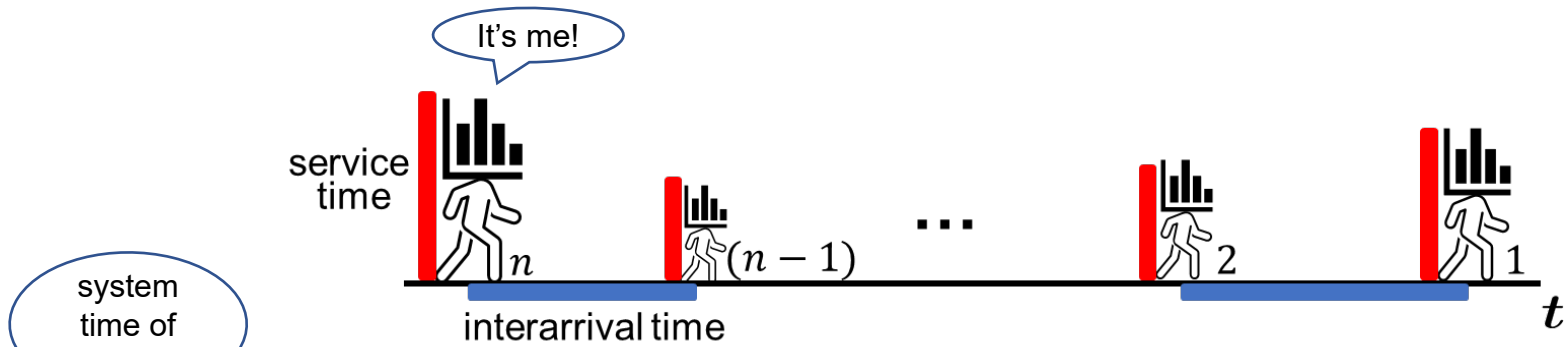
# Motivation of Uncertainty Set



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system time of update  $n$

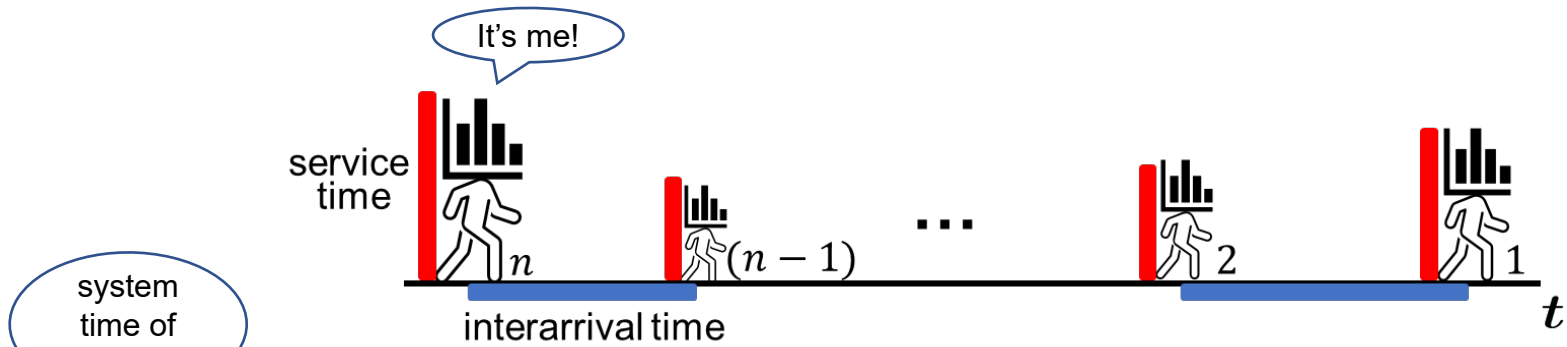
$$\text{grey bar} = \text{red bar} + \text{green bar}$$

$$\binom{n}{n} = \binom{n}{n} + \binom{n}{n}$$

$$\textcircled{1} \max_{1 \leq k \leq n} \left[ \binom{\text{red bars}}{\sum_k^n} - \binom{\text{blue bars}}{\sum_{k+1}^n} \right]$$

$\textcircled{1}$  D. V. Lindley, "The theory of queues with a single server," in Mathematical Proceedings of the Cambridge Philosophical Society

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$$\text{grey bar} = \text{red bar} + \text{green bar}$$

$$\binom{n}{n} = \binom{n}{n} + \binom{n}{n}$$

Sum of random variables?

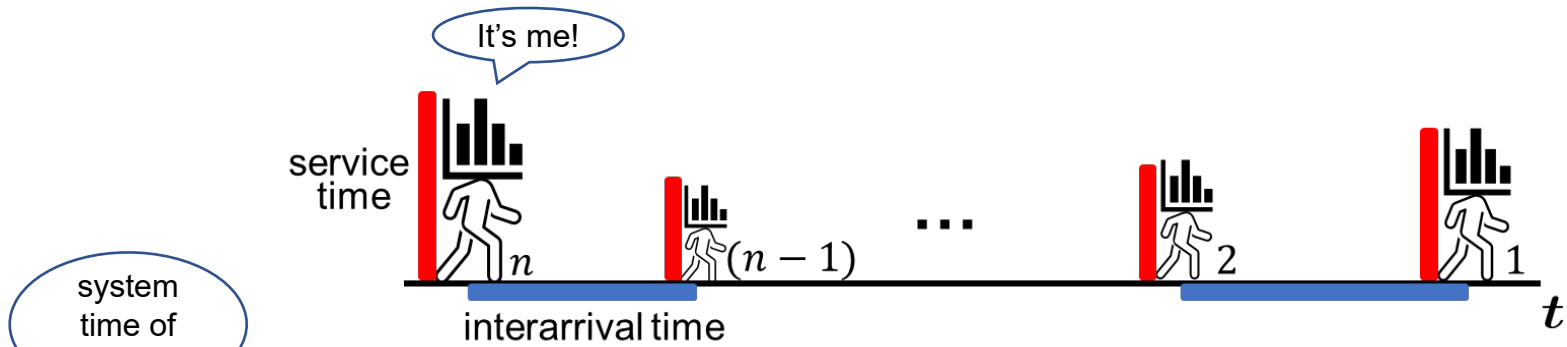
$$\textcircled{1} \max_{1 \leq k \leq n} \left[ \binom{n}{k} - \binom{n}{k+1} \right]$$

$$\textcircled{1} \max_{1 \leq k \leq n} \left[ \binom{n}{k_1} \dots \binom{n}{k_k} - \binom{n}{k_1+1} \dots \binom{n}{k_k+1} \right]$$

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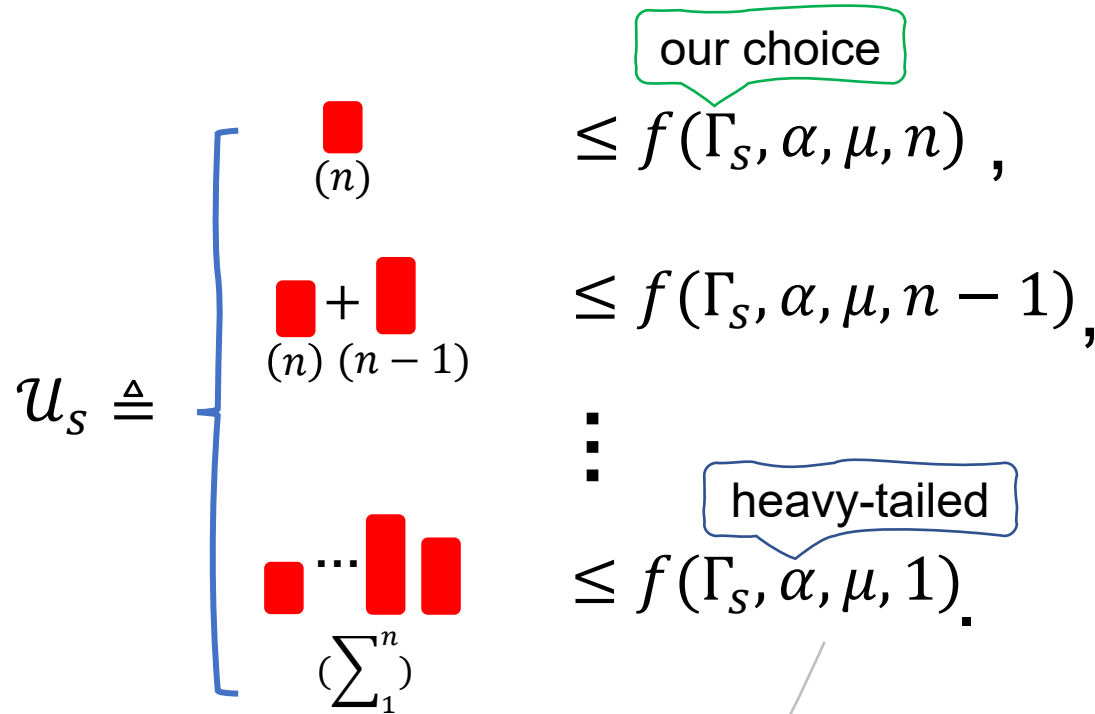
Generalized Central Limit Theorem!

*“sum of i.i.d. random variables converges to a stable distribution”*

$$\textcircled{1} \max_{1 \leq k \leq n} \left[ \begin{matrix} \left( \text{red bars}_{(n)} \right) \\ \left( \text{red bars}_{(n)} + \text{red bars}_{(n-1)} \right) - \left( \text{blue bars}_{(n)} \right) \\ \vdots \\ \left( \text{red bars}_{(\sum_k^n)} \right) - \left( \text{blue bars}_{(\sum_{k+1}^n)} \right) \\ \vdots \\ \left( \text{red bars}_{(\sum_1^n)} \right) - \left( \text{blue bars}_{(\sum_2^n)} \right) \end{matrix} \right]$$

<sup>①</sup> D. V. Lindley, “The theory of queues with a single server,” in Mathematical Proceedings of the Cambridge Philosophical Society

# Uncertainty Set of Service Time



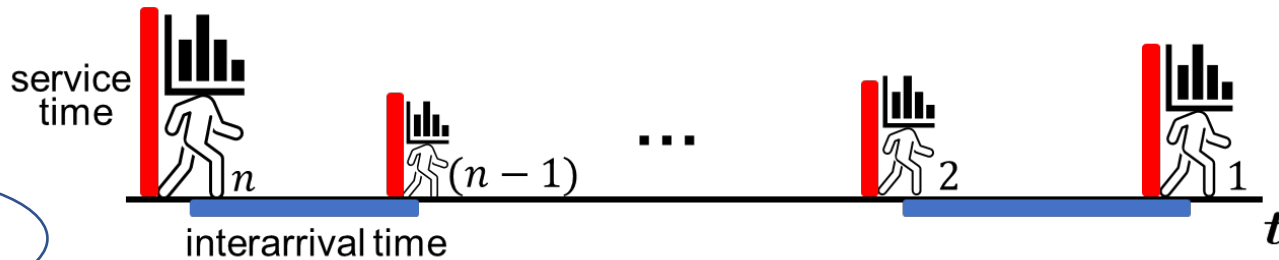
## Advantages :

- More general distributions
- Non-i.i.d. service process
- Heavy-tailed distributions

$$f(\Gamma_S, \alpha, \mu, k) = \Gamma_S(n - k + 1)^{\frac{1}{\alpha}} + (n - k + 1)/\mu$$



# Worst-Case System Time

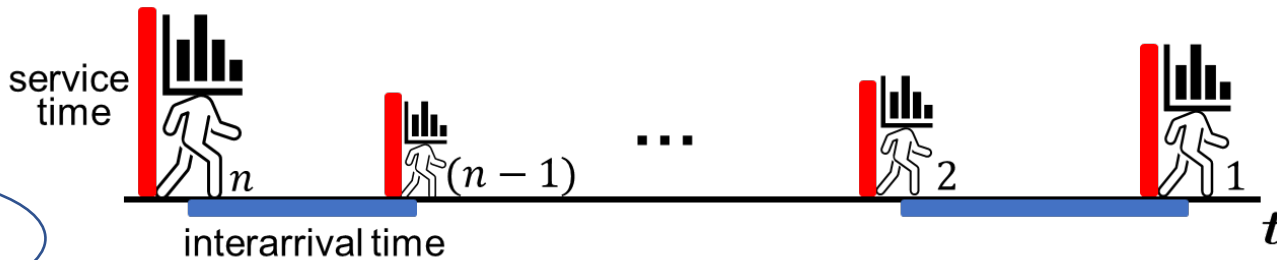


worst-case system time of update  $n$

$$\hat{\Delta}(n) \triangleq \max_{u_s} \max_{u_a} \max_{1 \leq k \leq n} \left[ \left( \sum_k^n \right) - \left( \sum_{k+1}^n \right) \right]$$

$$\leq \max_{1 \leq k \leq n} \left[ \max_{u_s} \left( \sum_k^n \right) - \min_{u_a} \left( \sum_{k+1}^n \right) \right]$$

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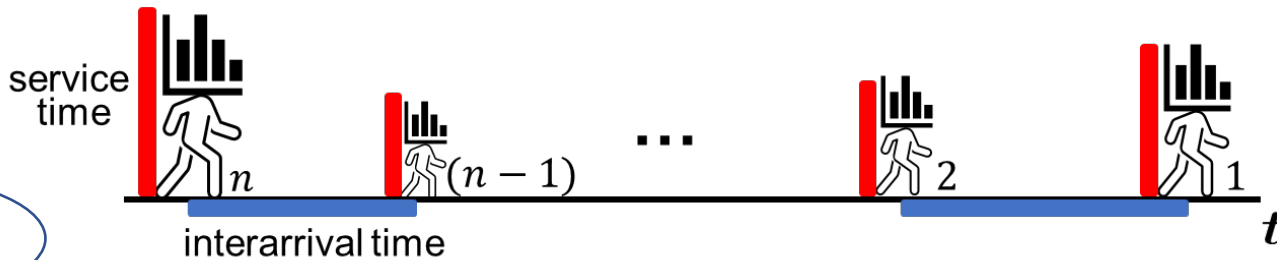
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$$= \max_{1 \leq k \leq n} \left[ \left( \hat{\Delta} \sum_k^n \right) - \left( \sum_{k+1}^n \hat{\Delta} \right) \right]$$

# Worst-Case System Time



worst-case system time of update  $n$

$$\hat{\Delta}(n) \triangleq \max_{u_s} \max_{u_a} \max_{1 \leq k \leq n} \left\{ \left[ \text{red bars } (\sum_k^n) - \text{blue bars } (\sum_{k+1}^n) \right] \right\}$$

$$\leq \max_{1 \leq k \leq n} \left\{ \max_{u_s} \left( \text{red bars } (\sum_k^n) \right) - \min_{u_a} \left( \text{blue bars } (\sum_{k+1}^n) \right) \right\}$$

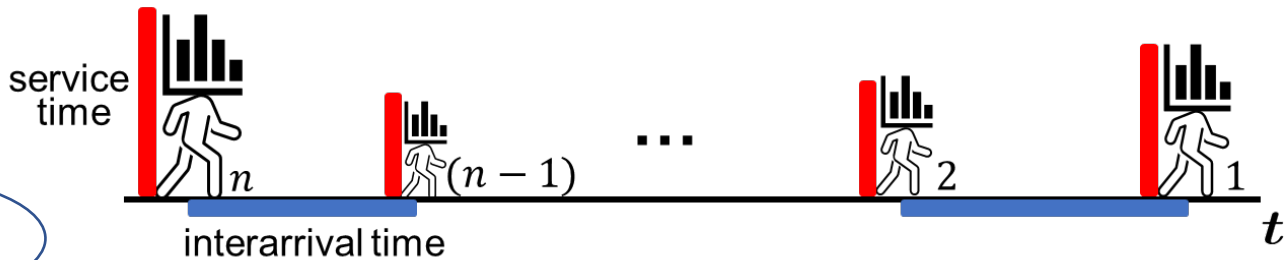
$f(\Gamma_s, \alpha, \mu, k)$

$g(\Gamma_a, \alpha, \lambda, k + 1)$

$$= \max_{1 \leq k \leq n} \left\{ \left[ \hat{\Delta} \text{ red bars } (\sum_k^n) - \hat{\Delta} \text{ blue bars } (\sum_{k+1}^n) \right] \right\}$$

$$(n - k + 1)/\mu - (n - k)/\lambda + \Gamma_s(n - k + 1)^{\frac{1}{\alpha}} + \Gamma_a(n - k)^{\frac{1}{\alpha}}$$

# Worst-Case System Time



worst-case system time of update  $n$

$$\hat{\Delta}(n) \triangleq \max_{u_s} \max_{u_a} \max_{1 \leq k \leq n} \left\{ \left[ \sum_{k+1}^n u_s - \sum_{k+1}^n u_a \right] \right\}$$

$$\leq \max_{1 \leq k \leq n} \left\{ \max_{u_s} \left[ \sum_{k+1}^n u_s \right] - \min_{u_a} \left[ \sum_{k+1}^n u_a \right] \right\}$$

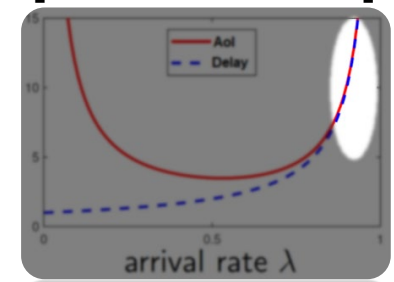
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$g(\Gamma_a, \alpha, \lambda, k+1)$

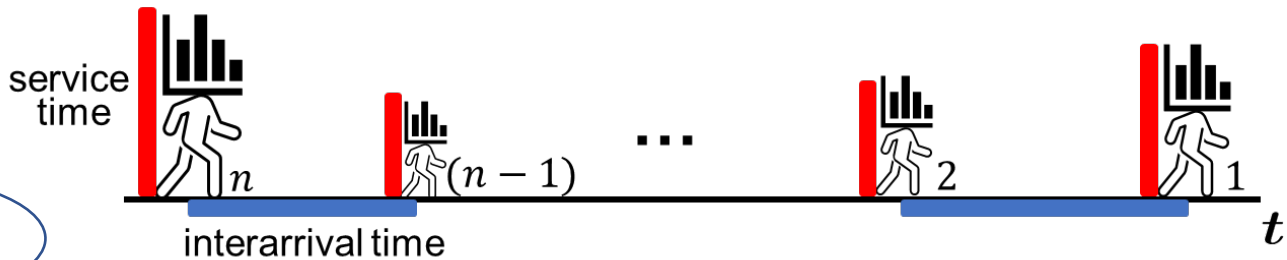
$$= \max_{1 \leq k \leq n} \left\{ \left[ \sum_{k+1}^n \hat{\Delta} \right] - \left[ \sum_{k+1}^n \hat{\Delta} \right] \right\}$$

$$(n-k+1)/\mu - (n-k)/\lambda + \Gamma_s(n-k+1)^{\frac{1}{\alpha}} + \Gamma_a(n-k)^{\frac{1}{\alpha}}$$

[Bandi et al. '18]



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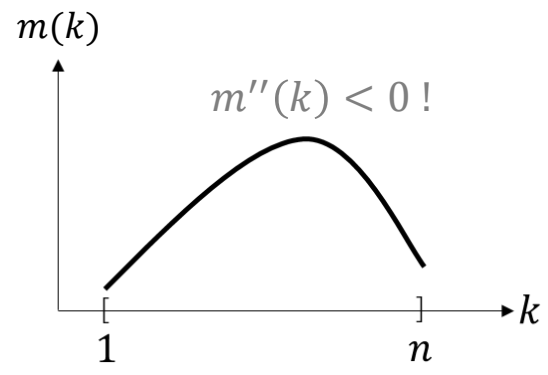
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$f(\Gamma_s, \alpha, \mu, k)$

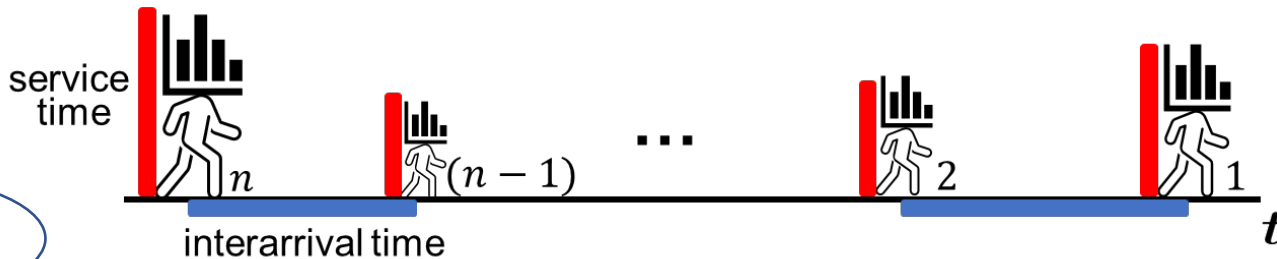
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$$m(k) = (n - k + 1)/\mu - (n - k)/\lambda + \Gamma_s(n - k + 1)^{\frac{1}{\alpha}} + \Gamma_a(n - k)^{\frac{1}{\alpha}}$$

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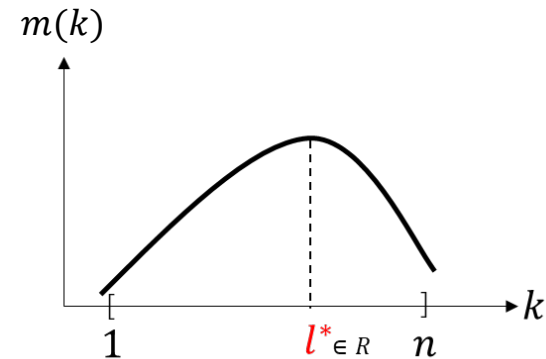
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$$f(\Gamma_s, \alpha, \mu, k)$$

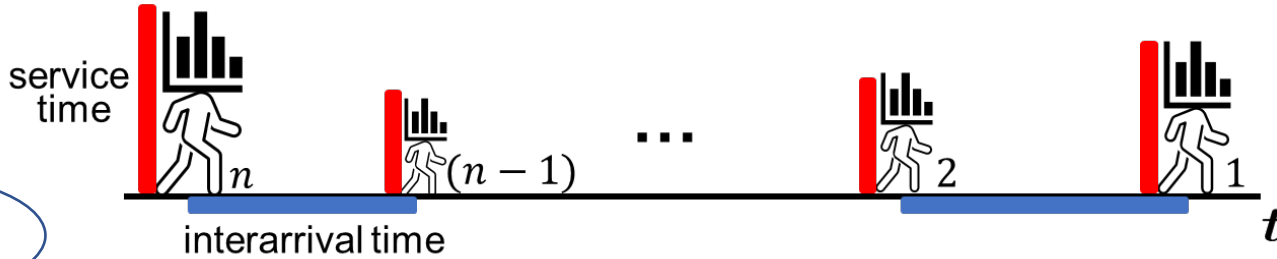
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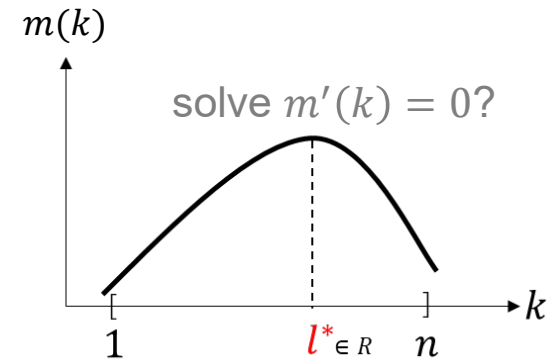
$$\hat{\Delta}(n) \triangleq \max_{u_s} \max_{u_a} \max_{1 \leq k \leq n} \left\{ \left[ \text{red bars } (\sum_k^n) - \text{blue bars } (\sum_{k+1}^n) \right] \right\}$$

$$\leq \max_{1 \leq k \leq n} \left\{ \max_{u_s} \left( \text{red bars } (\sum_k^n) \right) - \min_{u_a} \left( \text{blue bars } (\sum_{k+1}^n) \right) \right\}$$

$$f(\Gamma_s, \alpha, \mu, k)$$

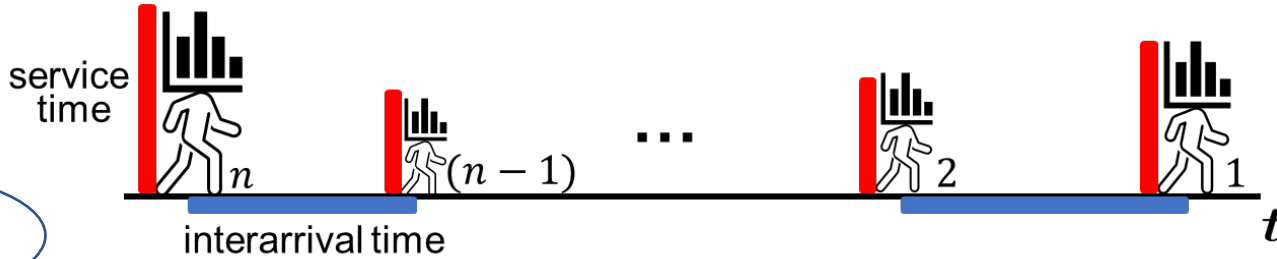
$$g(\Gamma_a, \alpha, \lambda, k+1)$$

$$= \max_{1 \leq k \leq n} \left\{ \left[ \hat{\Delta} \text{ red bars } (\sum_k^n) - \left[ \hat{\Delta} \text{ blue bars } (\sum_{k+1}^n) \right] \right] \right\}$$



$$m(k) = (n - k + 1)/\mu - (n - k)/\lambda + \Gamma_s(n - k + 1)^{\frac{1}{\alpha}} + \Gamma_a(n - k)^{\frac{1}{\alpha}}$$

# Worst-Case System Time



worst-case system time of update  $n$

$$\hat{\Delta}(n) \triangleq \max_{u_s} \max_{u_a} \max_{1 \leq k \leq n} \left[ \left( \sum_k^n \right) - \left( \sum_{k+1}^n \right) \right]$$

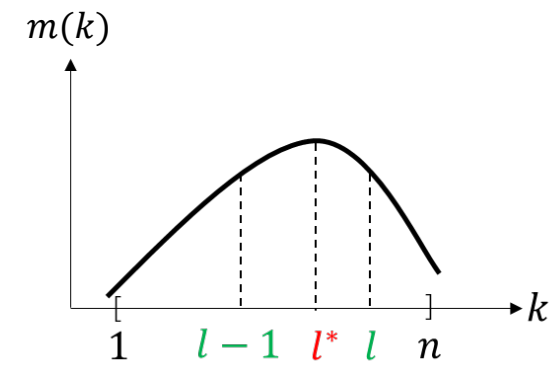
$$l = \left( \frac{\alpha(1/\lambda - 1/\mu)}{\Gamma_a + \Gamma_s} \right)^{\alpha/(1-\alpha)}$$

$$\leq \max_{1 \leq k \leq n} \left[ \max_{u_s} \left( \sum_k^n \right) - \min_{u_a} \left( \sum_{k+1}^n \right) \right]$$

$f(\Gamma_s, \alpha, \mu, k)$

$g(\Gamma_a, \alpha, \lambda, k+1)$

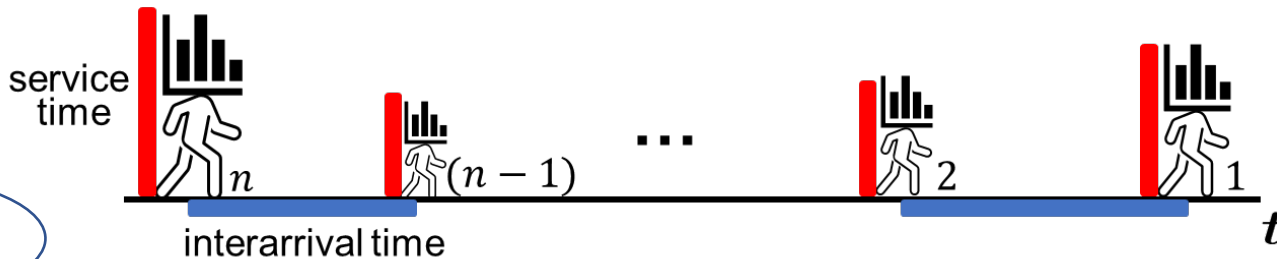
$$= \max_{1 \leq k \leq n} \left[ \left( \sum_k^n \right) - \left( \sum_{k+1}^n \right) \right]$$



$$m(k) = (n - k + 1)/\mu - (n - k)/\lambda + \Gamma_s(n - k + 1)^{\frac{1}{\alpha}} + \Gamma_a(n - k)^{\frac{1}{\alpha}}$$



# Worst-Case System Time



worst-case system time of update  $n$

$$\hat{\Delta}(n) \triangleq \max_{u_s} \max_{u_a} \max_{1 \leq k \leq n} \left[ \left( \sum_k^n \right) - \left( \sum_{k+1}^n \right) \right]$$

$$l = \left( \frac{\alpha(1/\lambda - 1/\mu)}{\Gamma_a + \Gamma_s} \right)^{\alpha/(1-\alpha)}$$

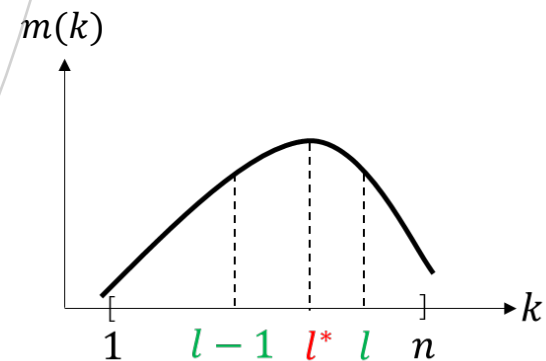
$$\leq \max_{1 \leq k \leq n} \left[ \max_{u_s} \left( \sum_k^n \right) - \min_{u_a} \left( \sum_{k+1}^n \right) \right]$$

$$f(\Gamma_s, \alpha, \mu, k)$$

$$g(\Gamma_a, \alpha, \lambda, k+1)$$

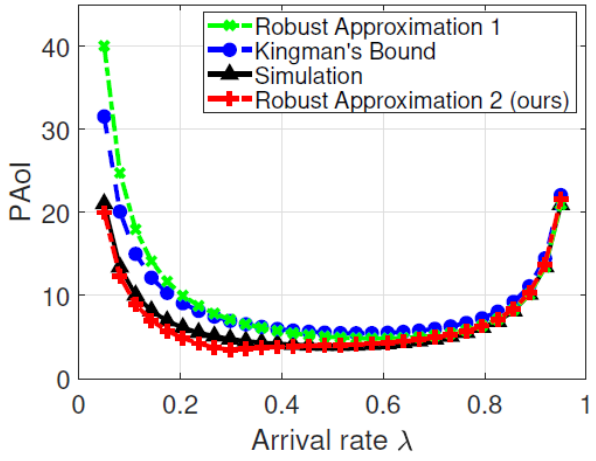
$$= \max_{1 \leq k \leq n} \left[ \left( \sum_k^n \right) - \left( \sum_{k+1}^n \right) \right]$$

integral maximizer  $k^* \in \mathbb{Z}^+ \in \{[l] - 1, [l], [l] + 1\}$

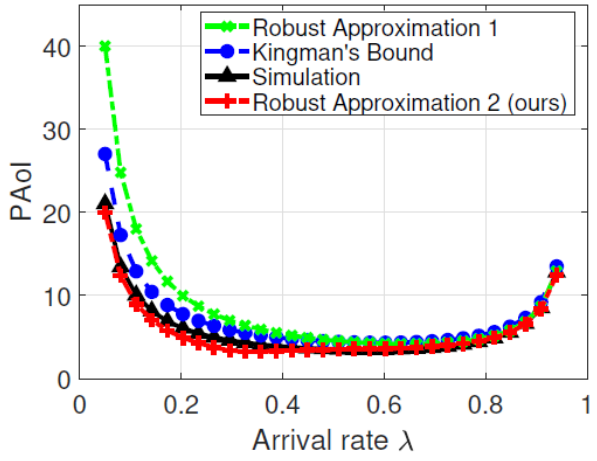


$$m(k) = (n-k+1)/\mu - (n-k)/\lambda + \Gamma_s(n-k+1)^{\frac{1}{\alpha}} + \Gamma_a(n-k)^{\frac{1}{\alpha}}$$

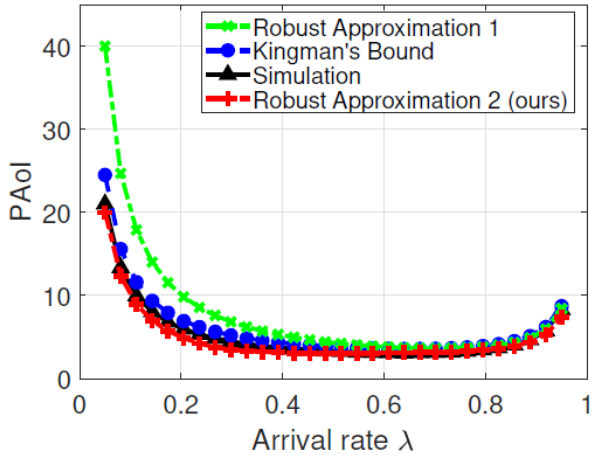
# Numerical Result (Single-Source)



(a) Exponential distribution



(b) Normal distribution

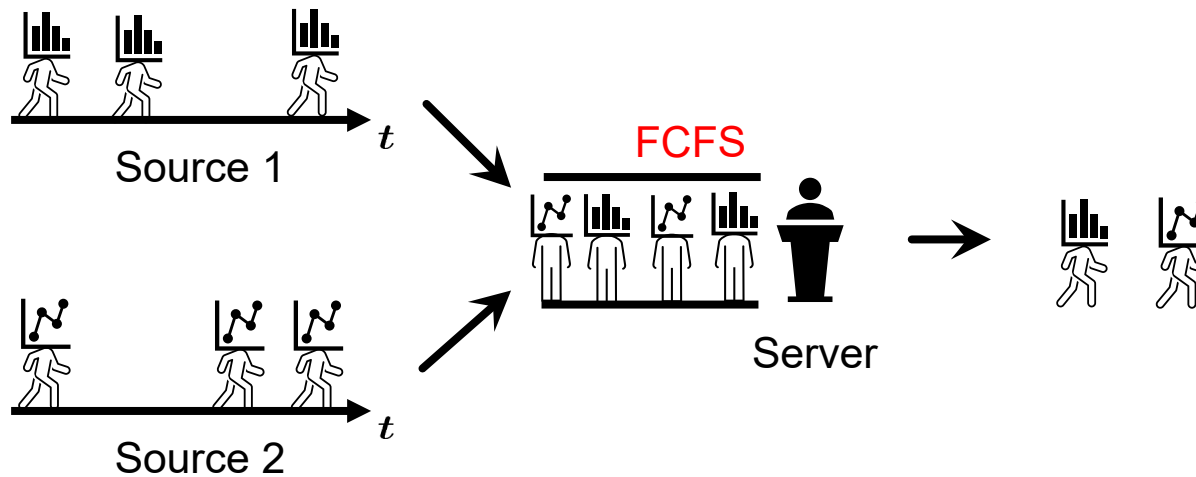


(c) Uniform distribution

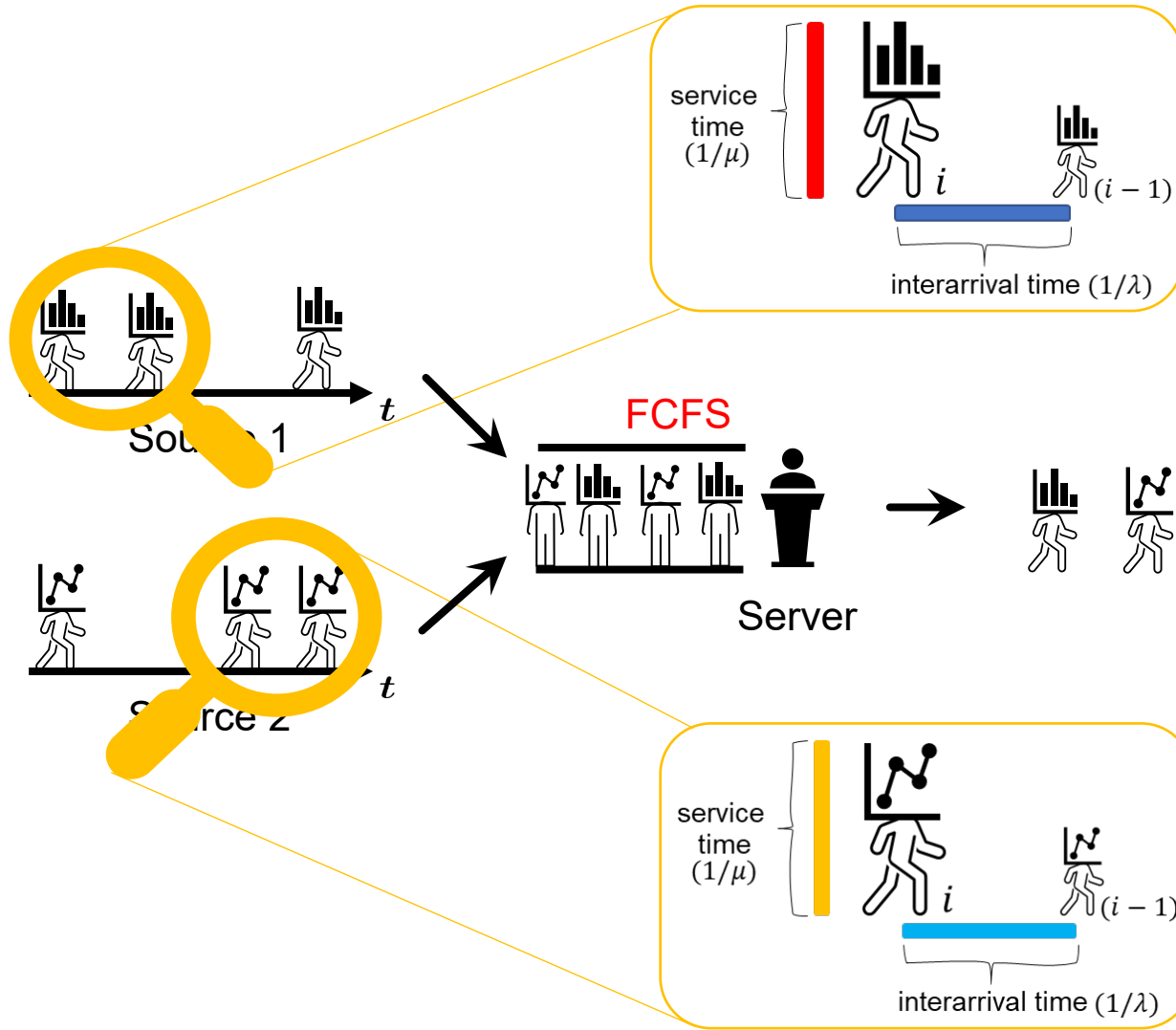
Methods	Exponential	Normal	Uniform
Kingman's bound	33.86%	22.58%	14.89%
Robust Approx. 1	32.01%	34.90%	36.49%
<b>Robust Approx. 2 (ours)</b>	<b>8.32 %</b>	<b>8.47 %</b>	<b>9.28 %</b>

Error percentage

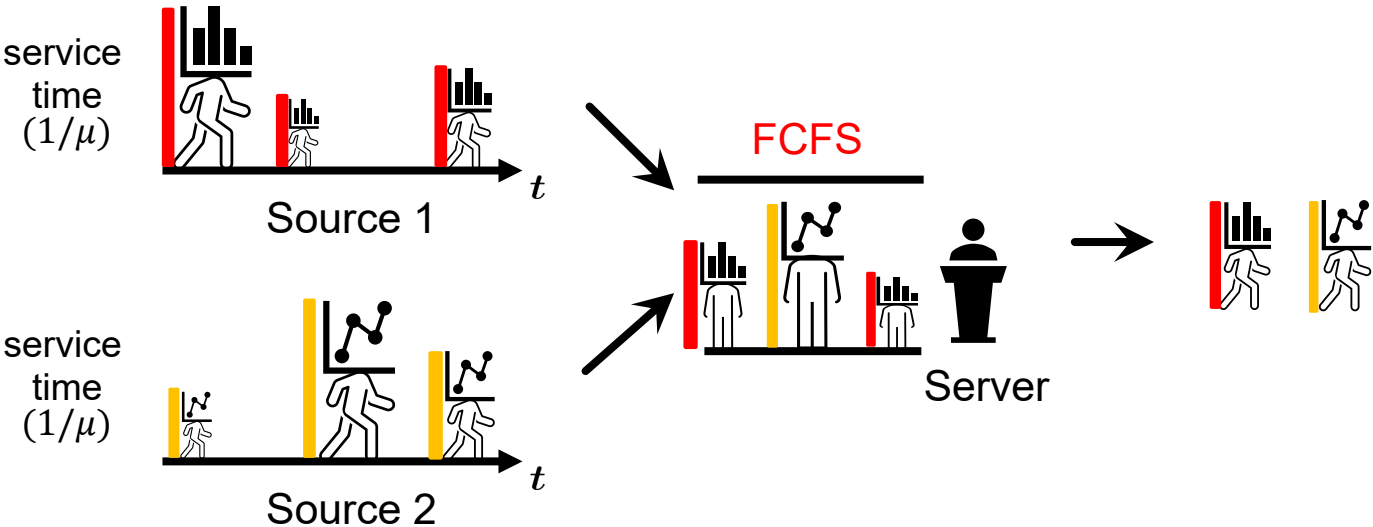
# Two-Source System



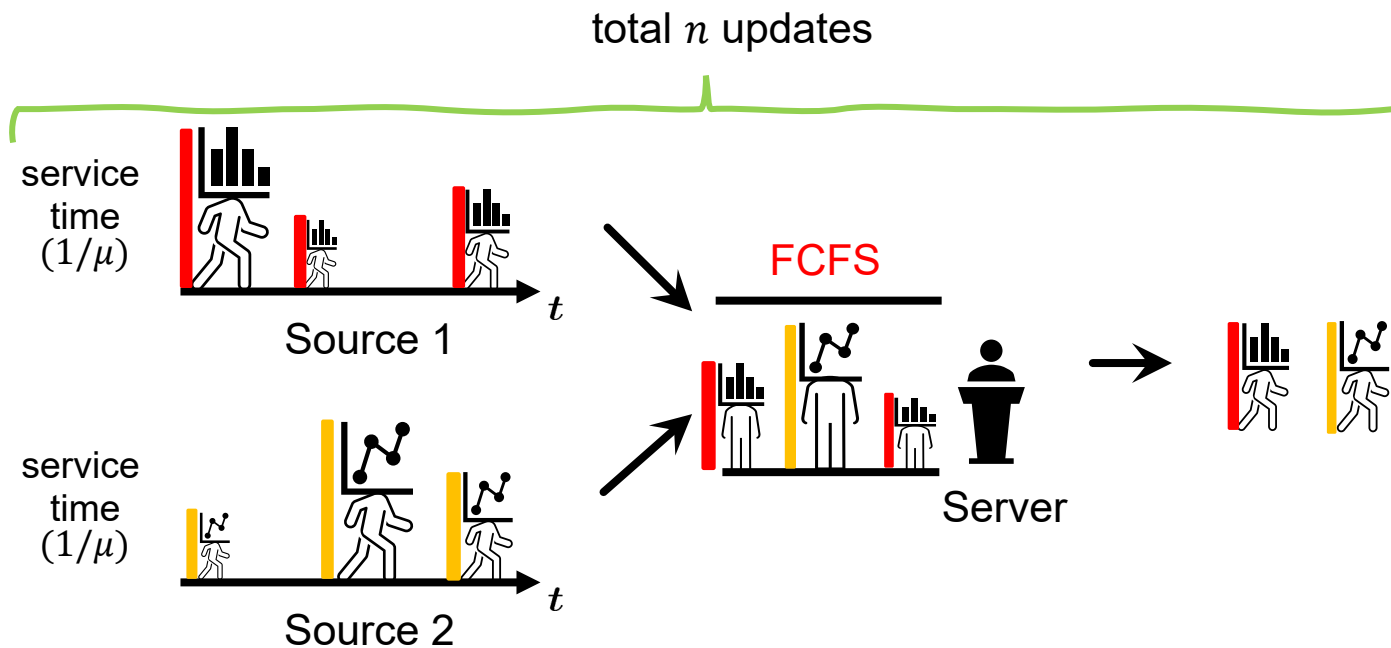
# Two-Source System



# Uncertainty Set of Service Time

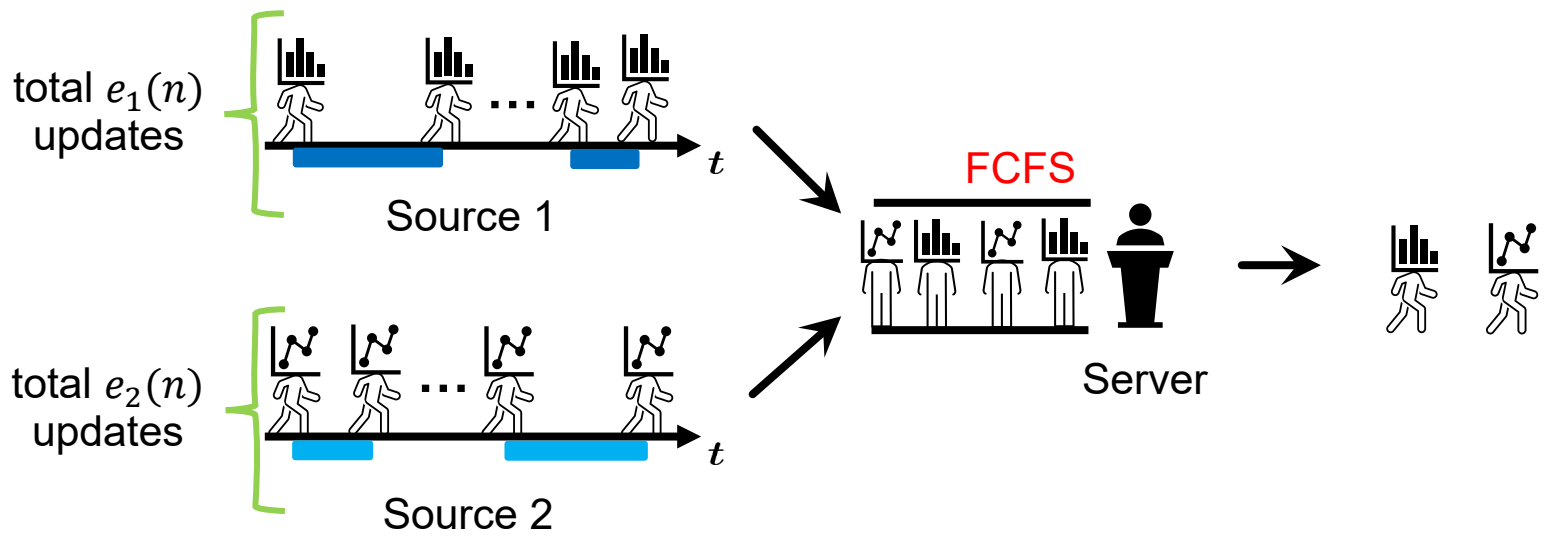


# Uncertainty Set of Service Time



$$\mathcal{U}_S \triangleq \left\{ \begin{array}{l} \begin{array}{c} \color{red}\blacksquare \\ (n) \end{array} \leq f(\Gamma_S, \alpha, \mu, n) \\ \begin{array}{c} \color{red}\blacksquare + \color{yellow}\blacksquare \\ (n) \quad (n-1) \end{array} \leq f(\Gamma_S, \alpha, \mu, n-1) \\ \vdots \\ \begin{array}{c} \color{red}\blacksquare \dots \color{yellow}\blacksquare \color{yellow}\blacksquare \\ (\sum_1^n) \end{array} \leq f(\Gamma_S, \alpha, \mu, 1) \end{array} \right.$$

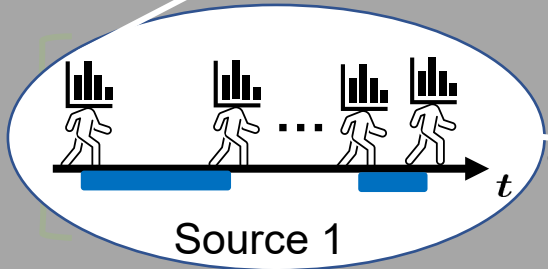
# Uncertainty Sets of Interarrival Time



$$e_1(n) + e_2(n) = n$$

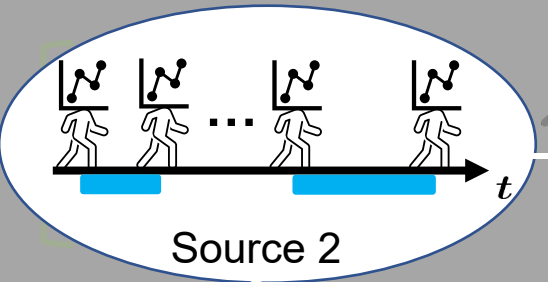
# Uncertainty Sets of Interarrival Time

total  $e_1(n)$  updates

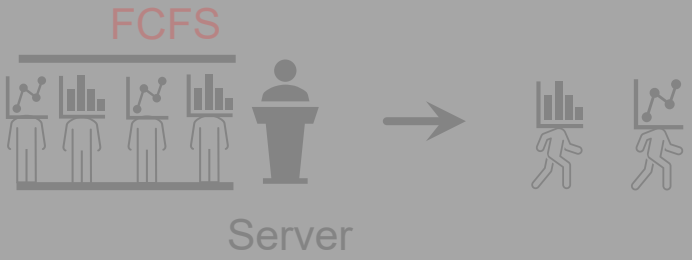


$$\mathcal{U}_a^1 \triangleq \begin{cases} \begin{matrix} \blacksquare & \geq g(\Gamma_a, \alpha, \lambda, e_1(n)) \\ (e_1(n)) \end{matrix} \\ \begin{matrix} \blacksquare + \blacksquare & \geq g(\Gamma_a, \alpha, \lambda, e_1(n) - 1) \\ (e_1(n)) & (e_1(n) - 1) \end{matrix} \\ \vdots \\ \begin{matrix} \blacksquare \cdots \blacksquare & \geq g(\Gamma_a, \alpha, \lambda, 1) \\ (\sum_1^{e_1(n)}) \end{matrix} \end{cases}$$

total  $e_2(n)$  updates

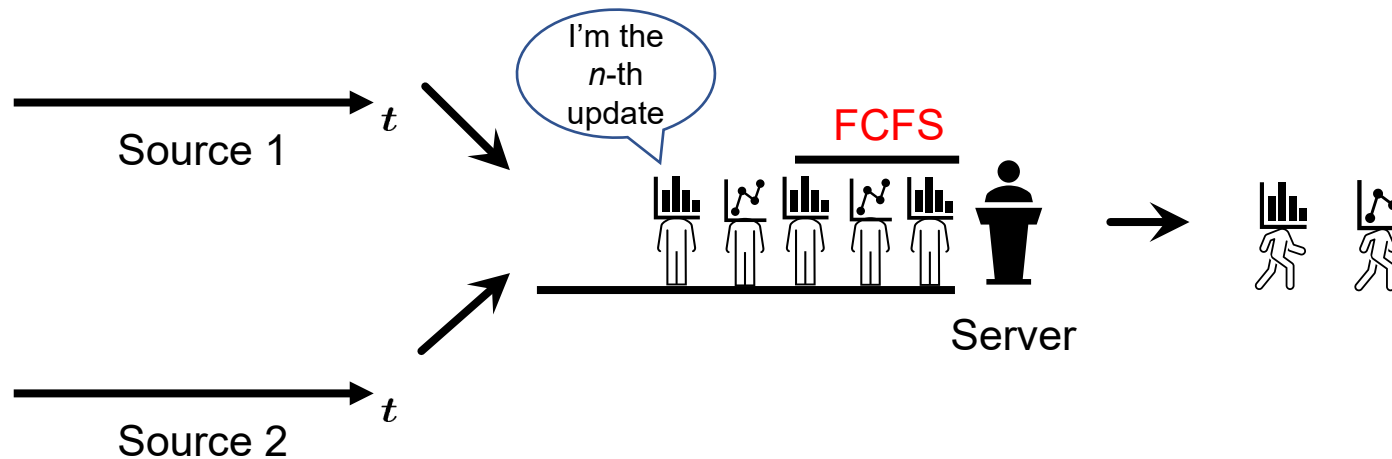


$$\mathcal{U}_a^2 \triangleq \begin{cases} \begin{matrix} \blacksquare & \geq g(\Gamma_a, \alpha, \lambda, e_2(n)) \\ (e_2(n)) \end{matrix} \\ \begin{matrix} \blacksquare + \blacksquare & \geq g(\Gamma_a, \alpha, \lambda, e_2(n) - 1) \\ (e_2(n)) & (e_2(n) - 1) \end{matrix} \\ \vdots \\ \begin{matrix} \blacksquare \cdots \blacksquare & \geq g(\Gamma_a, \alpha, \lambda, 1) \\ (\sum_1^{e_2(n)}) \end{matrix} \end{cases}$$





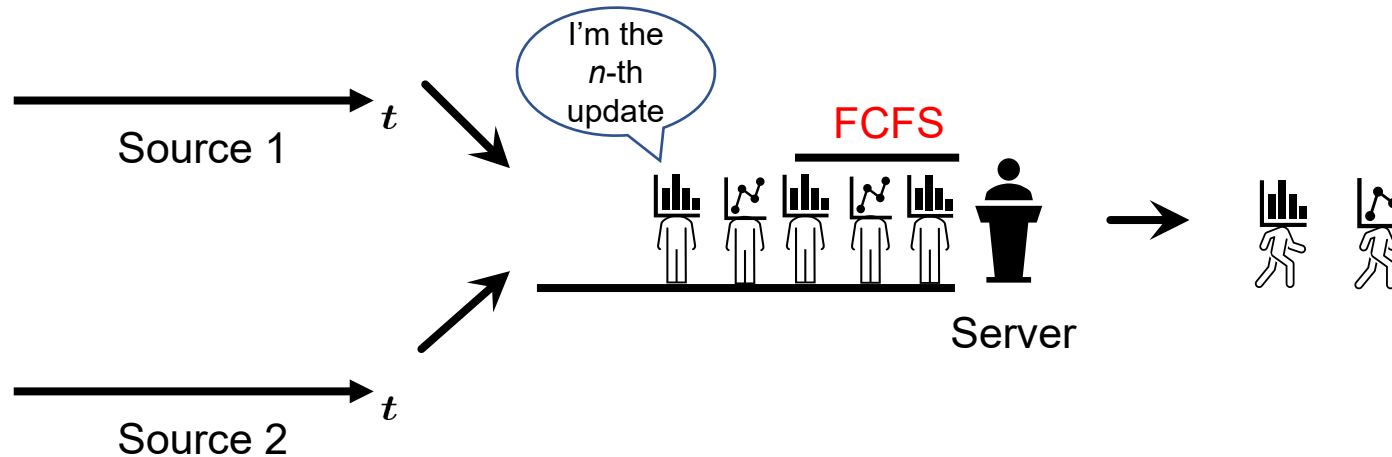
# Worst-Case System Time



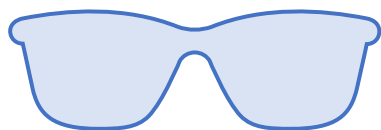
worst-case system time of update  $n$  on the sample path

$$\leq \max_{1 \leq k \leq n} \left[ \max \left( \begin{array}{c} \text{?} \\ \left[ \begin{array}{c} n \cdots k \\ \text{sum of} \\ \text{service time} \end{array} \right] \end{array} \right) - \min \left( \begin{array}{c} \text{?} \\ \left[ \begin{array}{c} n \cdots k+1 \\ \text{sum of} \\ \text{interarrival} \\ \text{time} \end{array} \right] \end{array} \right) \right]$$

# Worst-Case System Time

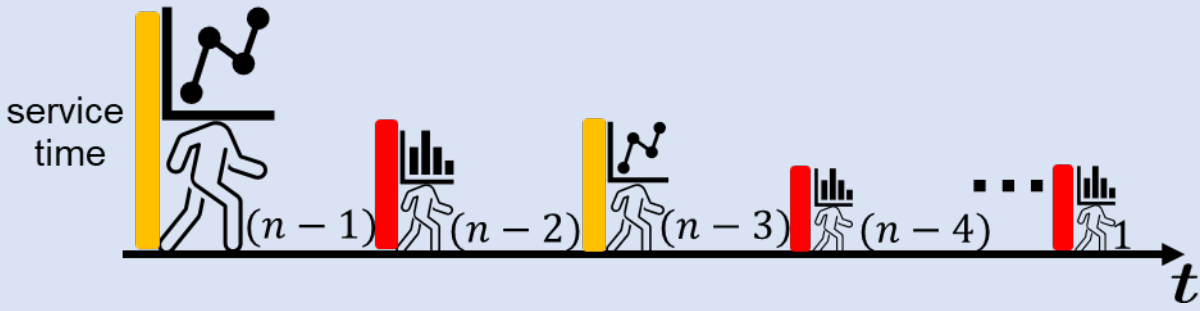


$$\text{worst-case system time of update } n \text{ on the sample path} \leq \max_{1 \leq k \leq n} \left[ \max \left( \begin{array}{c} \text{sum of service time} \\ \left[ \begin{array}{c} n \dots k \end{array} \end{array} \right] \right) - \min \left( \begin{array}{c} \text{sum of interarrival time} \\ \left[ \begin{array}{c} n \dots k+1 \end{array} \end{array} \right] \right) \right]$$



point of view of  $n$ -th update on whole sample path

# Sum of Service Time

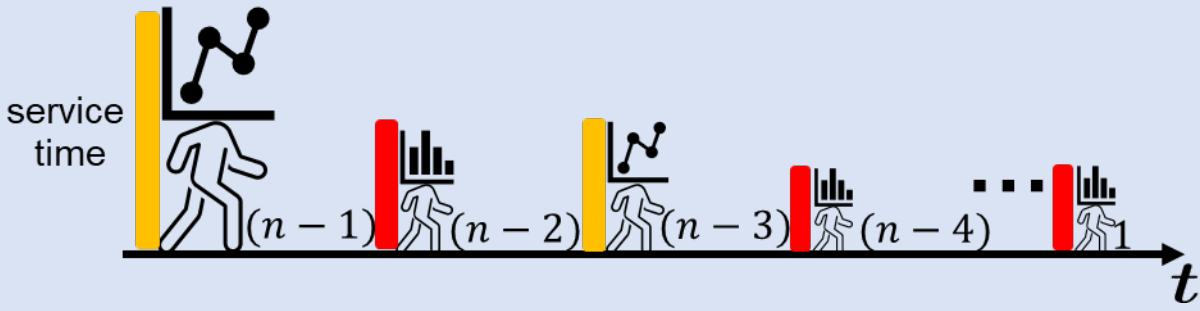


$$\mathcal{U}_s \triangleq \begin{cases} \text{red bar } (n) & \leq f(\Gamma_s, \alpha, \mu, n) \\ \text{red bar } (n) + \text{yellow bar } (n-1) & \leq f(\Gamma_s, \alpha, \mu, n-1) \\ \vdots & \\ \text{red bar } \dots \text{yellow bars } (\sum_1^n) & \leq f(\Gamma_s, \alpha, \mu, 1) \end{cases}$$

max  $\left( \begin{matrix} \boxed{n} & \dots & \boxed{k} \\ \text{sum of} \\ \text{service time} \end{matrix} \right)$

point of view of  $n$ -th update on service times of whole sample path

# Sum of Service Time



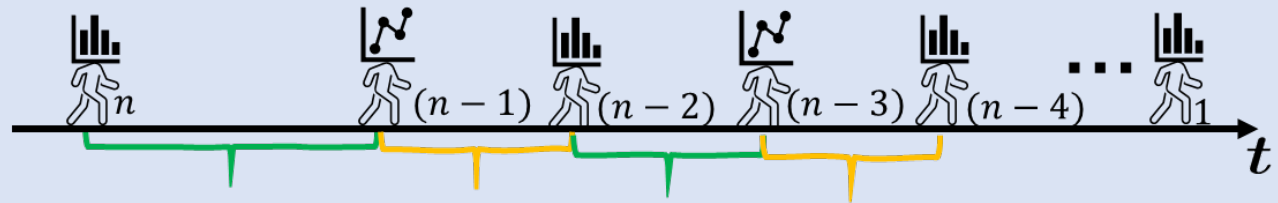
$$\mathcal{U}_s \triangleq \begin{cases} \text{red bar } (n) & \leq f(\Gamma_s, \alpha, \mu, n) \\ \text{red bar } (n) + \text{yellow bar } (n-1) & \leq f(\Gamma_s, \alpha, \mu, n-1) \\ \vdots & \\ \text{red bar } (n) + \dots + \text{yellow bar } (1) & \leq f(\Gamma_s, \alpha, \mu, 1) \end{cases}$$

$$\max \left( \begin{matrix} \text{green checkmark} \\ n \dots k \\ \text{sum of} \\ \text{service time} \end{matrix} \right)$$

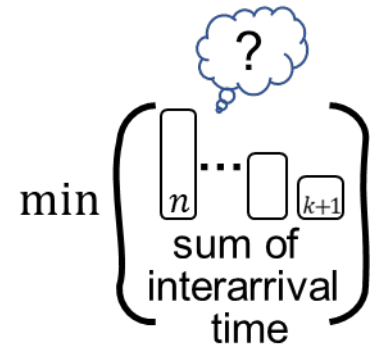
point of view of  $n$ -th update  
on service times of whole sample path

# Challenge: Sum of Interarrival Time

VIRGINIA TECH.



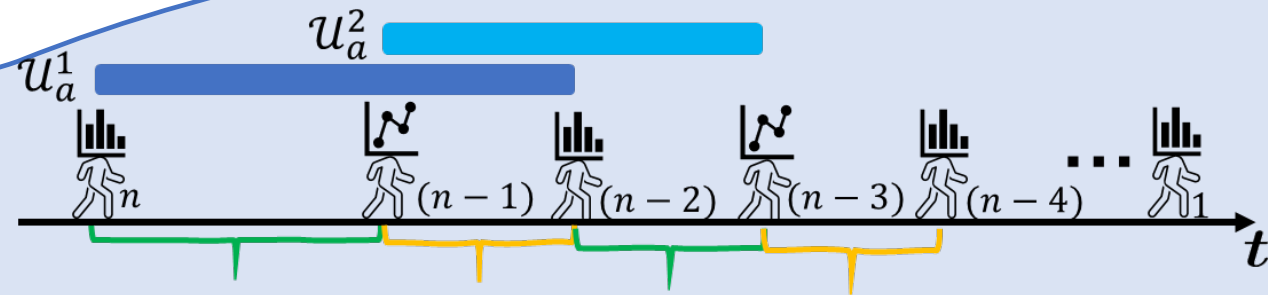
interarrival times of updates experienced **on the whole sample path!**



point of view of  $n$ -th update  
on interarrival times of whole sample path

# Challenge: Sum of Interarrival Time

VIRGINIA TECH.



interarrival times of updates experienced **on the whole sample path!**

$$U_a^1 \triangleq \begin{cases} (e_1(n)) & \geq g(\Gamma_a, \alpha, \lambda, e_1(n)) \\ (e_1(n)) + (e_1(n) - 1) & \geq g(\Gamma_a, \alpha, \lambda, e_1(n) - 1) \\ \vdots \\ (\sum_1^{e_1(n)}) & \geq g(\Gamma_a, \alpha, \lambda, 1) \end{cases}$$

$$U_a^2 \triangleq \begin{cases} (e_2(n)) & \geq g(\Gamma_a, \alpha, \lambda, e_2(n)) \\ (e_2(n)) + (e_2(n) - 1) & \geq g(\Gamma_a, \alpha, \lambda, e_2(n) - 1) \\ \vdots \\ (\sum_1^{e_2(n)}) & \geq g(\Gamma_a, \alpha, \lambda, 1) \end{cases}$$

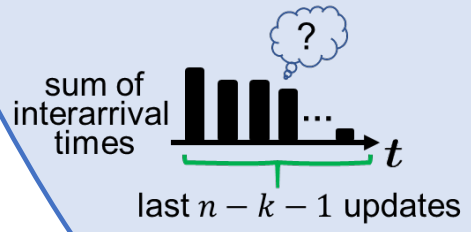
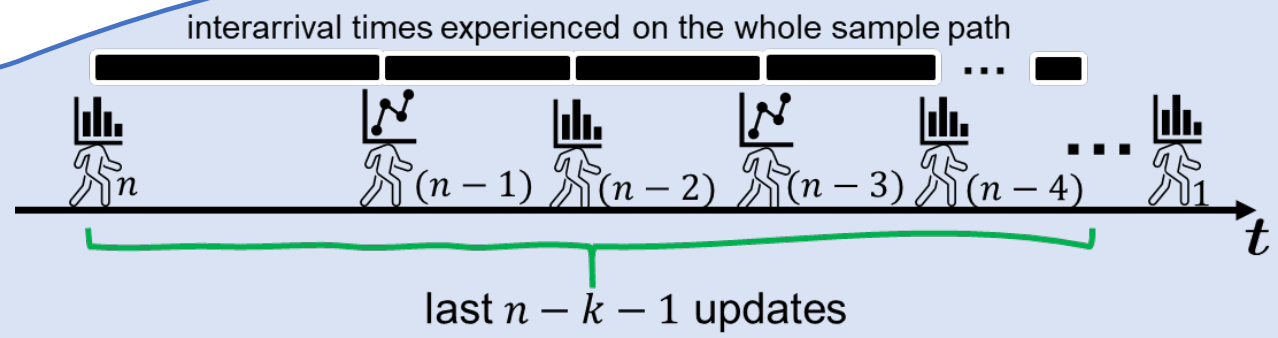


our uncertainty sets of interarrival time are defined for **each source**

min  $\left( \begin{matrix} n & \dots & k+1 \\ \text{sum of} \\ \text{interarrival} \\ \text{time} \end{matrix} \right)$

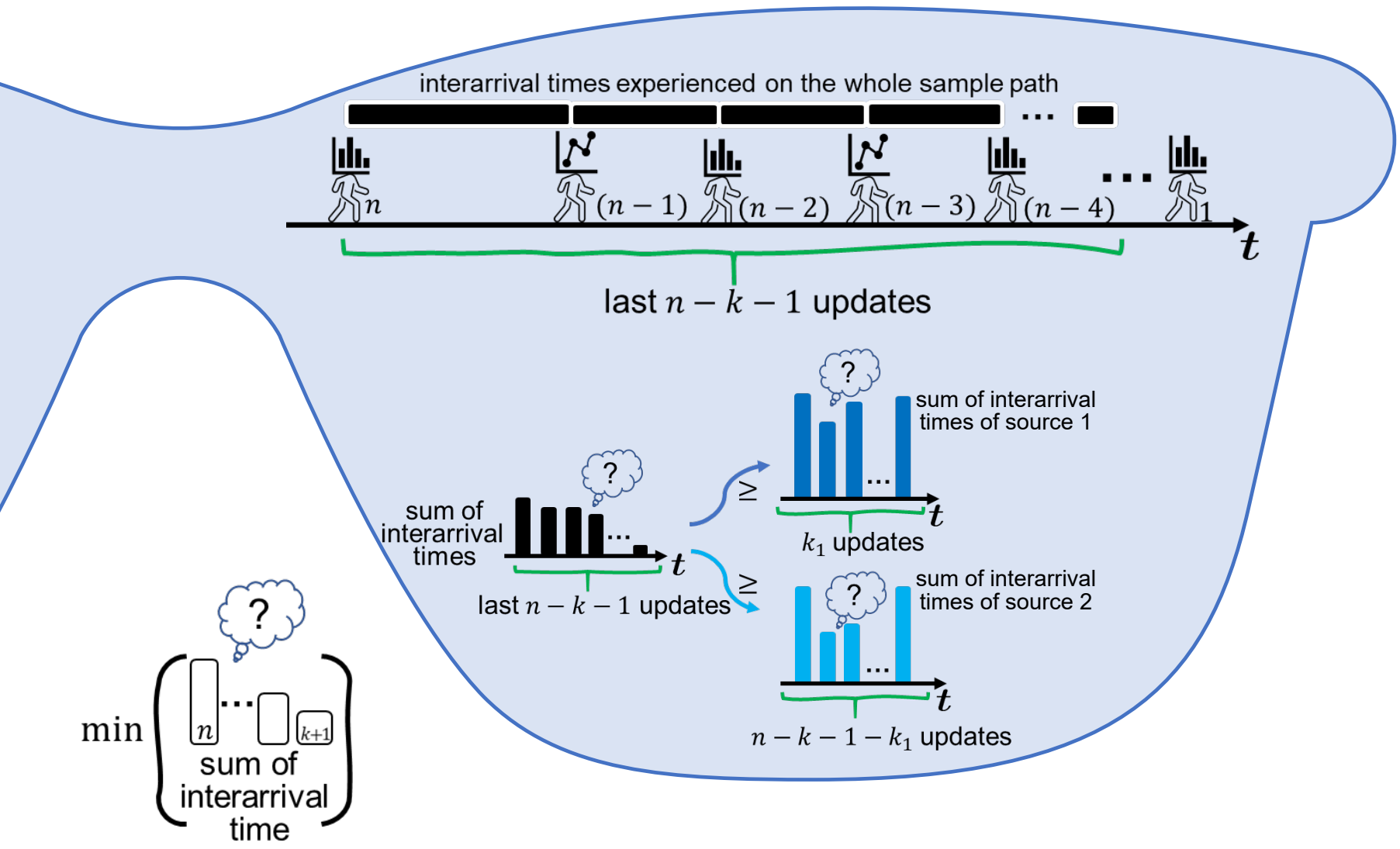
point of view of  $n$ -th update  
on interarrival times of whole sample path

# Our Solution



min  $\left( \begin{array}{c} \text{?} \\ \text{sum of} \\ \text{interarrival} \\ \text{time} \end{array} \right)$

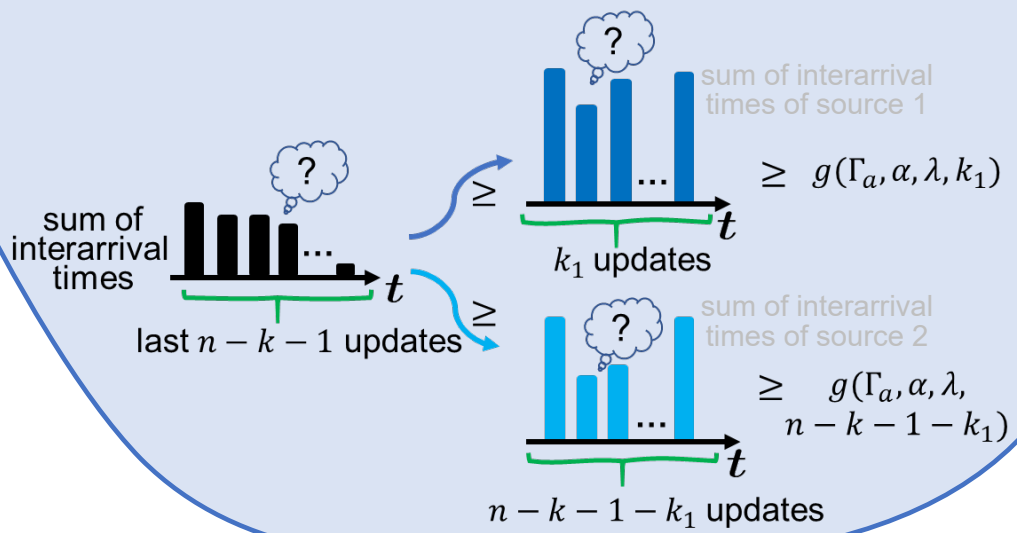
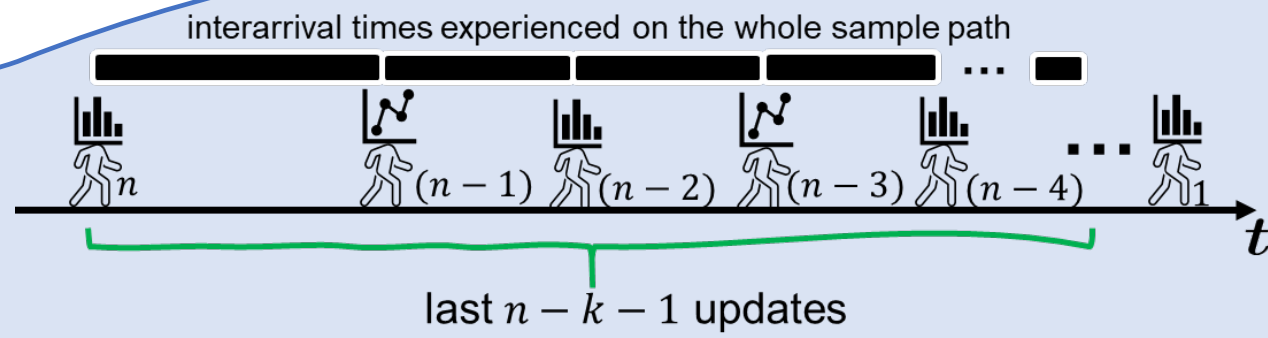
# Our Solution



$$\min \left( \begin{array}{c} \text{?} \\ \left[ \begin{array}{c} \text{bar } n \\ \dots \\ \text{bar } k+1 \end{array} \right] \\ \text{sum of} \\ \text{interarrival} \\ \text{time} \end{array} \right)$$

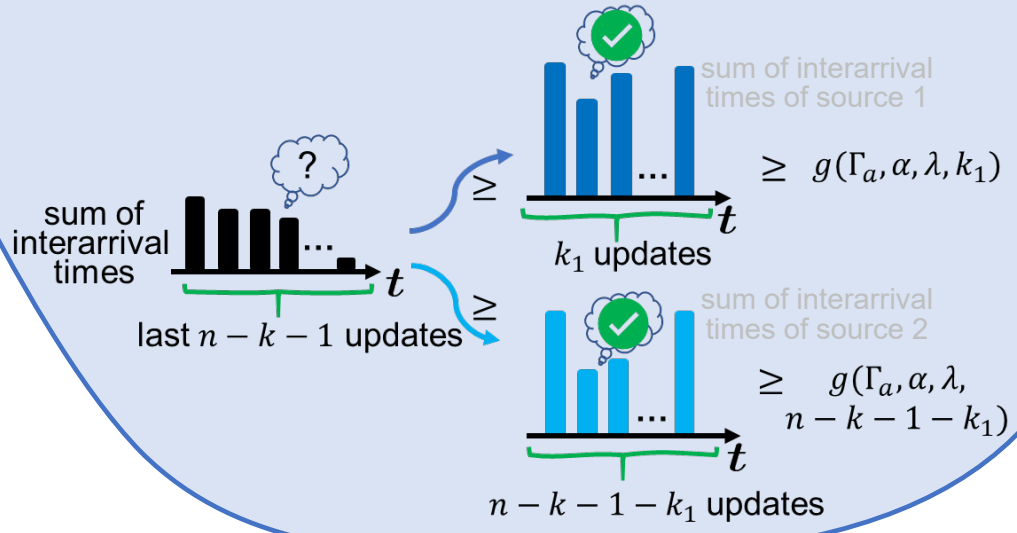
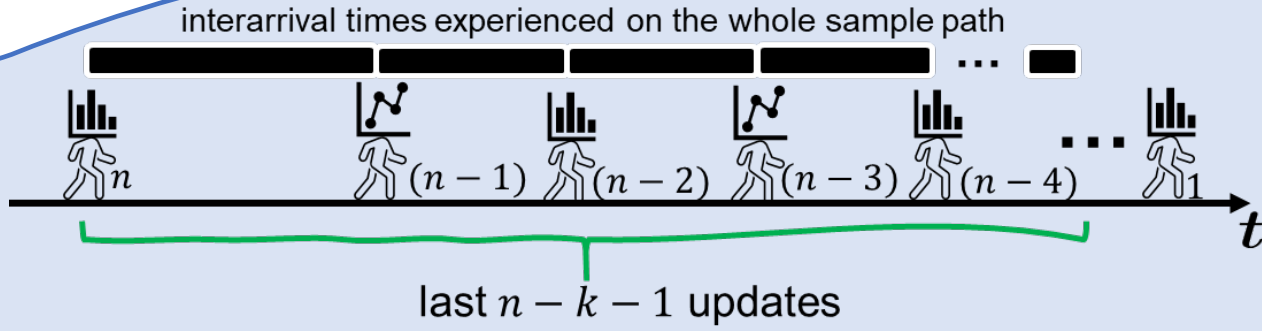


# Our Solution



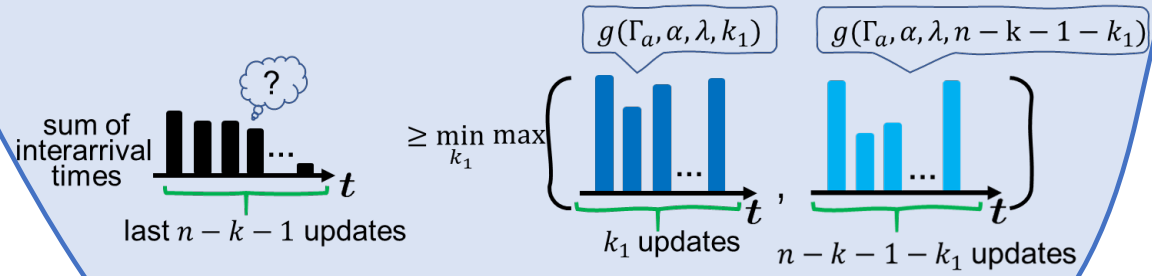
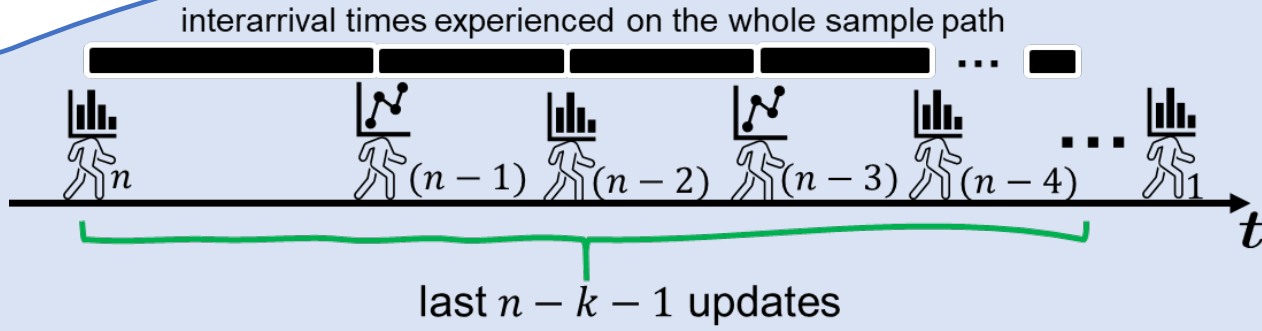
$$\min \left( \begin{array}{c} \text{?} \\ \left[ \begin{array}{c} n \\ \dots \\ k+1 \end{array} \right] \\ \text{sum of} \\ \text{interarrival} \\ \text{time} \end{array} \right)$$

# Our Solution



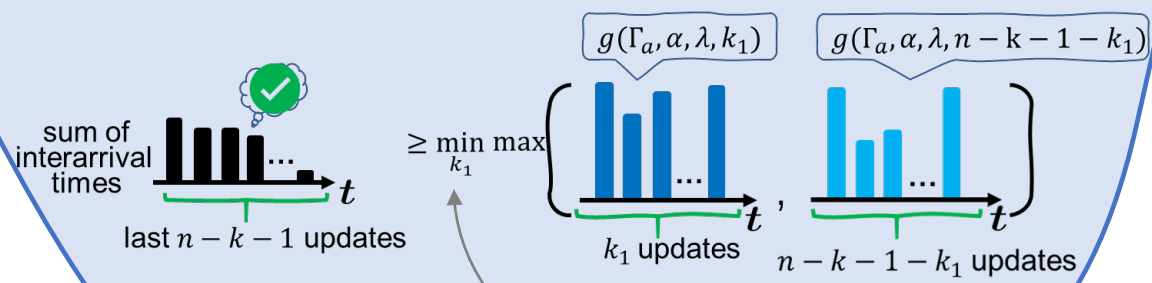
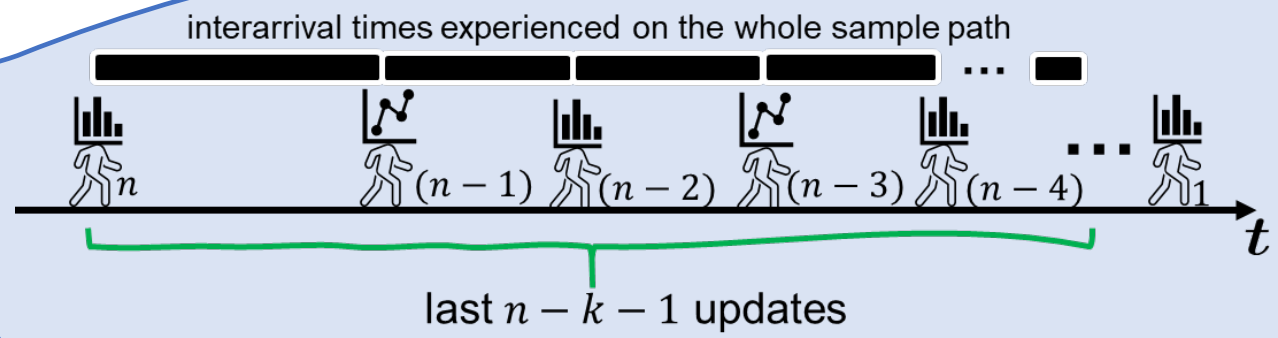
min  $\left( \begin{matrix} \text{?} \\ n \dots k+1 \\ \text{sum of} \\ \text{interarrival} \\ \text{time} \end{matrix} \right)$

# Our Solution



min  $\left( \begin{matrix} \text{?} \\ n \dots k+1 \\ \text{sum of} \\ \text{interarrival} \\ \text{time} \end{matrix} \right)$

# Our Solution

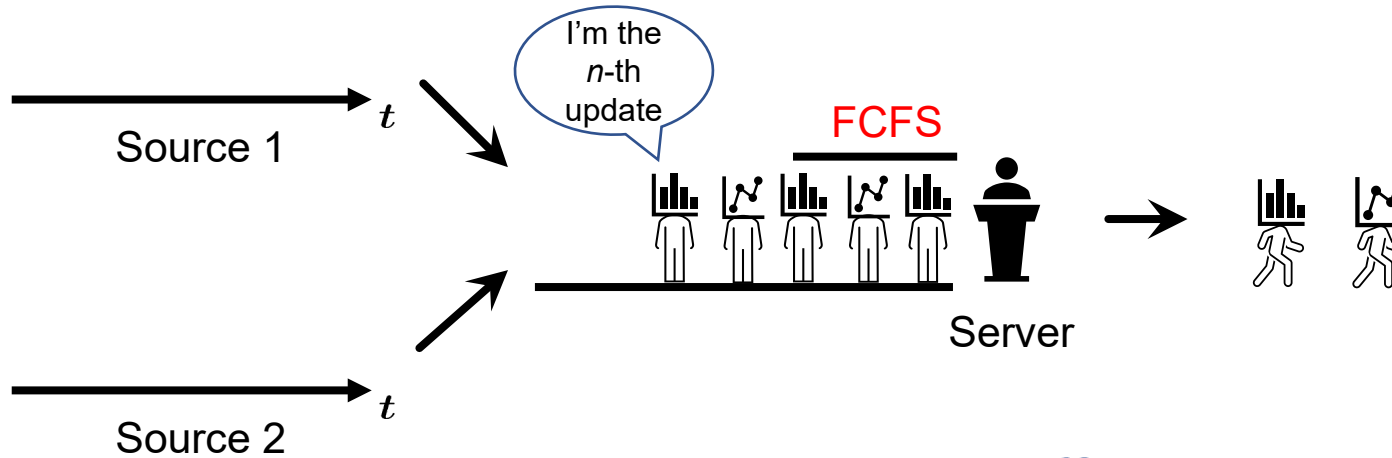


$$\geq \min_{k_1} \max$$

$$(n - k - 1)/2\lambda + \Gamma_a((n - k - 1)/2)\alpha^{\frac{1}{2}}$$

$$\min \left( \begin{matrix} \text{sum of} \\ \text{interarrival} \\ \text{time} \end{matrix} \right)$$

# Worst-Case System Time

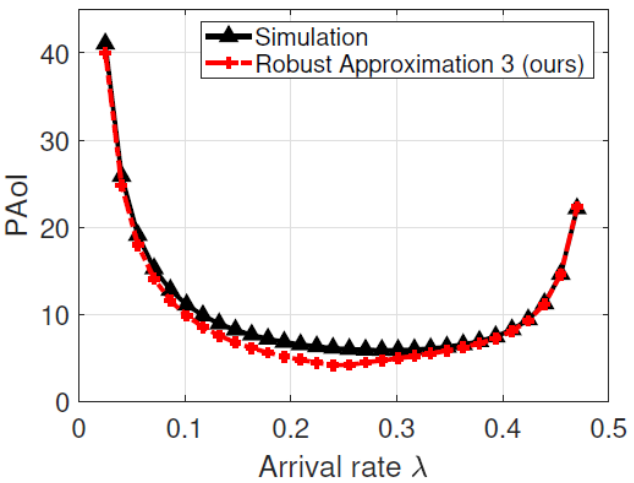


worst-case system time of update  $n$  on the sample path

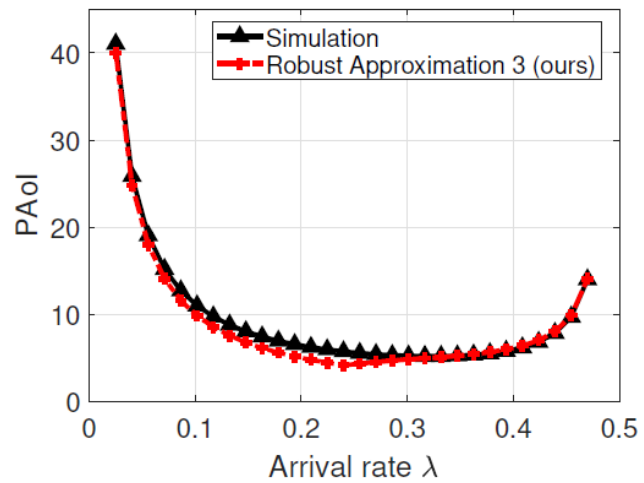
$$\leq \max_{1 \leq k \leq n} \left[ \max \left( \begin{array}{c} \text{sum of} \\ \text{service time} \end{array} \right) - \min \left( \begin{array}{c} \text{sum of} \\ \text{interarrival} \\ \text{time} \end{array} \right) \right]$$

$$(n - k + 1)/\mu - (n - k - 1)/2\lambda + \Gamma_s(n - k + 1)^{\frac{1}{\alpha}} + \Gamma_a((n - k - 1)/2)^{\frac{1}{\alpha}}$$

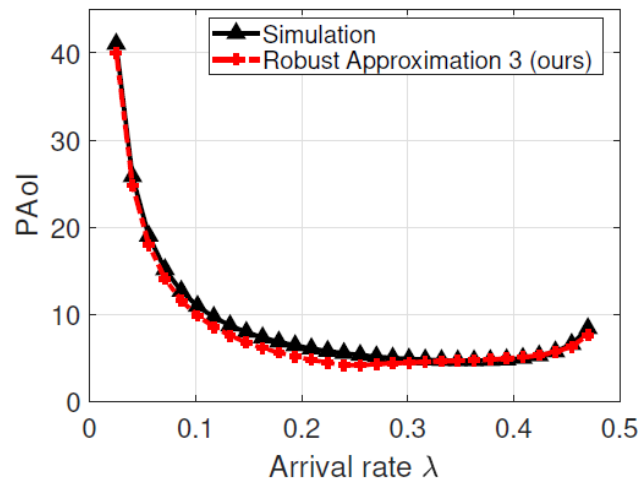
# Numerical Result (Two-Source)



(a) Exponential distribution



(b) Normal distribution



(c) Uniform distribution

Methods	Exponential	Normal	Uniform
<b>Robust Approx. 3 (ours)</b>	<b>12.68%</b>	<b>10.05%</b>	<b>9.79%</b>

Error percentage

- Applied robust queueing theory to analyzing PAol
  - Uncertainty sets
  - Worst-case analysis
- Single-source system
  - New robust bound of PAol
- Two-source system
  - Resolve new challenges
  - Robust bound of PAol

- Applied robust queueing theory to analyzing PAol
  - Uncertainty sets
  - Worst-case analysis
- Single-source system
  - New robust bound of PAol
- Two-source system
  - Resolve new challenges
  - Robust bound of PAol



- Multiple-source system
- Asymmetric sources
- Heterogeneous tail coefficients (i.e.,  $\alpha$ )
- Dependence of arrival/service processes



**Thank You!**

*Questions?*

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